On the Relationships between Period and Cohort Fertility

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1. Introduction

The relationships between period and cohort fertility measures have been one of the central issues in demography. In this regard, Ryder (1960, 1964) was one the first pioneers in trying to establish the relationships mathematically, who also coined the term "demographic translation" for the process. Since then, many researchers have made contributions in terms of improving and extending the original translation equations developed by Ryder (e.g. Foster, 1990; Ní Bhrolcháin, 1992; Bongaarts and Feeney, 1998; Keilman, 2000; Kohler and Philipov, 2001; Zeng and Land, 2002; Rodríguez, 2006), and a number of important results have been achieved.

However, there has been, to some extent, a lack of a unified analytical framework, which connects the dots in a more systematic way. This paper attempts to further examine the quantitative relationships between period and cohort fertility based on a unified analytical framework.

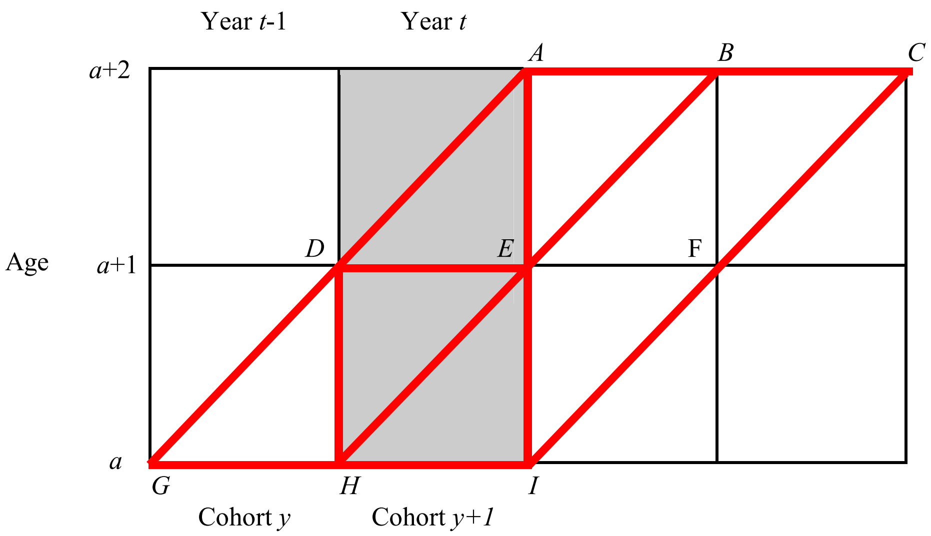
2. A general relationship between period and cohort fertility

In demography, age, period and cohort are three key dimensions and the Lexis diagram is a powerful tool to facilitate age-period-cohort (A-P-C) analysis. The Lexis diagram provides a graphical representation of the relationships among age, period and cohort. Figure 1 shows a portion of the Lexis diagram, in which we have the following correspondences:

- *Cohort-age analysis* (i.e. cohort y and age a) corresponds to parallelogram *DGHE*, which crosscuts two years (i.e. $t-1$ and t) and is called the cohort-age parallelogram.
- *Period-cohort analysis* (i.e. year t and cohort y) corresponds to parallelogram *DHEA*, which crosscuts two ages (i.e. a and $a+1$) and is called the period-cohort parallelogram.
- *Period-age analysis* (i.e. year t and age a) corresponds to square *DHIE*, which crosscuts two cohorts (i.e. cohort y and cohort $y+1$) and is called the period-age square.
- *Age-period-cohort analysis* (i.e. age a, year t, and cohort y) corresponds to triangle *DHE*, which is called the age-period-cohort triangle.

When discussing measures of demographic events (e.g. fertility), it is very important to be clear about which geometric shape is addressed. For example, the conventional (period) total fertility rate (*TFR*) is defined as the sum (total) of the age-specific fertility rates based on the squares (i.e. the period-age squares) of the corresponding year.

Figure 1. Lexis diagram for age-period-cohort analysis

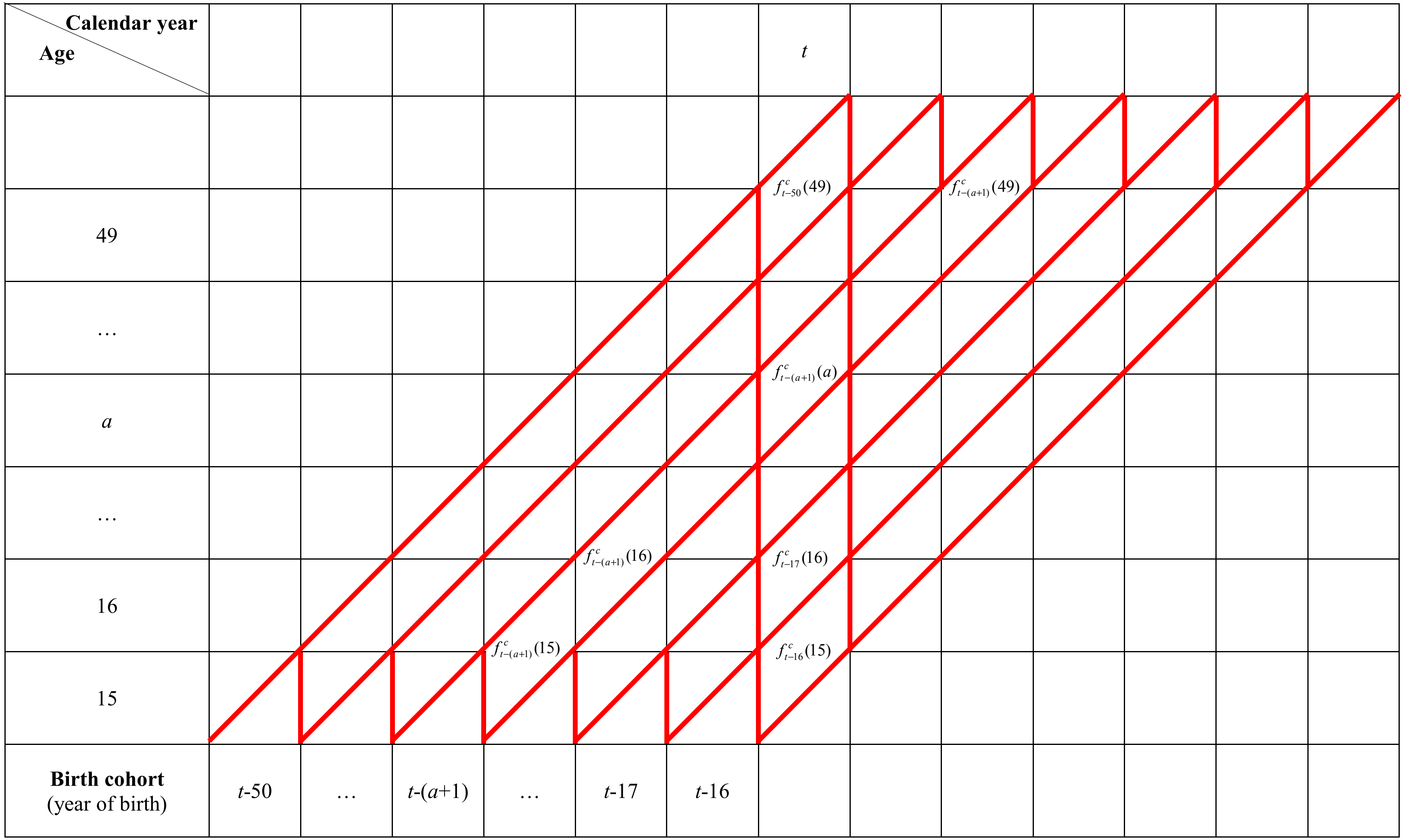


Since the focus of this paper is on the relationships between period and cohort fertility, the geometric shape to be used is the period-cohort parallelogram (i.e. *DHEA* in Figure 1). Therefore, for the purpose of this paper, the Lexis diagram in Figure 2 is used as a unified analytical framework for examining the quantitative relationships between period and cohort fertility. When Figure 2 is read horizontally, it relates to age analysis (i.e. age-period, or age-cohort); when Figure 2 is read vertically, it relates to period analysis (i.e. period-age, or period-cohort); when Figure 2 is read diagonally, it relates to cohort analysis (i.e. cohort-age, or cohort-period).

Suppose that year t is the period under study. From Figure 2, it is obvious that women who are aged a at the beginning of year t must be born in year $t-a-1$.[1] Now we define a few variables as follows. Let W_y^c represent the number of women who were born in year y (y = t-50, t-49, …, t-16), where the superscript c stands for cohort. Obviously, W_y^c constitute a birth cohort. For women of birth cohort y (i.e. diagonal in Figure 2), let $W_y^c(a)$ represent the number of women who are aged a (a = 15, 16, …, 49) at the beginning of the corresponding year. We assume that there is no mortality before the end of women's reproductive lifespan, then we have $W_y^c(15) = W_y^c(16) = \cdots = W_y^c(49) = W_y^c$.

[1] Please note that in this paper, age a always refers to the age of women at the beginning of a corresponding year.

Figure 2. Lexis diagram – A unified analytical framework



Let B_y^c denote the total number of live births that W_y^c delivered during their entire reproductive lifespan and $B_y^c(a)$ denote the number of live births that W_y^c delivered during the year at the beginning of which the women were aged *a*. Here, $B_y^c(a)$ corresponds to the parallelogram concerned. It is obvious that $B_y^c = \sum_{a=15}^{49} B_y^c(a)$.

For birth cohort *y*, we define its (cohort) age-specific fertility rates as follows:

$$f_y^c(a) = B_y^c(a) \big/ W_y^c(a)\,, \quad a = 15, 16, \ldots, 49 \qquad (2.1)$$

Please note that the $f_y^c(a)$ defined above are actually cohort-period measures, i.e. they are based on the cohort-period parallelogram (i.e. *DHEA* in Figure 1), not the cohort-age parallelogram (i.e. *DGHE* in Figure 1).

For birth cohort *y*, we define its (cohort) lifetime fertility rate (*LFR*) as the average number of live births that W_y^c delivered during their entire reproductive lifespan, i.e. $LFR_y = B_y^c \big/ W_y^c$. Then, we have

$$LFR_y = \frac{\sum_{a=15}^{49} B_y^c(a)}{W_y^c} = \sum_{a=15}^{49} \frac{B_y^c(a)}{W_y^c} = \sum_{a=15}^{49} \frac{B_y^c(a)}{W_y^c(a)} = \sum_{a=15}^{49} f_y^c(a) \qquad (2.2)$$

Equation (2.2) indicates that, for each birth cohort (*y*), its lifetime fertility rate equals the sum of the cohort age-specific fertility rates.[2]

For birth cohort *y*, the sequence $\left\{ f_y^c(a) \,\middle|\, a = 15, 16, \ldots, 49 \right\}$ constitutes an age distribution of the lifetime fertility rate of the cohort (i.e. LFR_y). For the sequence $\left\{ f_y^c(a) \,\middle|\, a = 15, 16, \ldots, 49 \right\}$, we define its standardized age pattern (schedule) of fertility as $\left\{ h_y^c(a) \,\middle|\, a = 15, 16, \ldots, 49 \right\}$, where $h_y^c(a) = f_y^c(a) \big/ LFR_y$, $a = 15, 16, \ldots, 49$. It is obvious that $h_y^c(a) \geq 0 \quad (a = 15, 16, \ldots, 49)$ and $\sum_{a=15}^{49} h_y^c(a) = 1$. From the definition above, we see that for birth cohort *y*, $h_y^c(a)\%$ of the LFR_y was born in year $y + a + 1$, at the beginning of which, the women were aged *a*. From the above definitions, we obtain $h_y^c(a) = B_y^c(a) \big/ B_y^c$, $a = 15, 16, \ldots, 49$. Furthermore, we have

$$f_y^c(a) = h_y^c(a) \cdot LFR_y, \quad a = 15, 16, \ldots, 49 \tag{2.3}$$

Equation (2.3) indicates that $f_y^c(a)$ can be expressed as the product of two cohort factors, one is a cohort fertility level factor (i.e. LFR_y), the other is a cohort fertility timing factor (i.e. $h_y^c(a)$).[3] Therefore, any cohort or period fertility measures based on $f_y^c(a)$ will have a level component and a timing component.

Now let's look at the Lexis diagram in Figure 2 from a period perspective. It is obvious that, for year *t*, we have two period curves of age-specific fertility rates, i.e. $\{ f_{t-(a+1)}^c(a) \mid a = 15, 16, \ldots, 49 \}$ and $\{ h_{t-(a+1)}^c(a) \mid a = 15, 16, \ldots, 49 \}$. For convenience hereafter, we denote $f_t^p(a) = f_{t-(a+1)}^c(a)$ and $h_t^p(a) = h_{t-(a+1)}^c(a)$, $a = 15, 16, \ldots, 49$, where the superscript *p* stands for period. Then we have

$$f_t^p(a) = h_t^p(a) \cdot LFR_{t-(a+1)}, \quad a = 15, 16, \ldots, 49 \tag{2.4}$$

Now, we define the (period) total fertility rate (*TFR*) for year *t* as follows:

$$TFR_t = f_{t-16}^c(15) + f_{t-17}^c(16) + \cdots + f_{t-50}^c(49) = \sum_{a=15}^{49} f_{t-(a+1)}^c(a) = \sum_{a=15}^{49} f_t^p(a) \tag{2.5}$$

Please note that the definition of the total fertility rate of year *t* (i.e. TFR_t) above is based on the age-specific fertility rates that correspond to the concerned cohort-period parallelograms, while the conventional *TFR* is based on the age-specific fertility rates that correspond to the concerned period-age squares.

From equations (2.4) and (2.5), we have

$$TFR_t = \sum_{a=15}^{49} f_t^p(a) = \sum_{a=15}^{49} [h_t^p(a) \cdot LFR_{t-(a+1)}] \tag{2.6}$$

Define $G_t = \sum_{a=15}^{49} h_t^p(a)$, then equation (2.6) can be rewritten as

$$TFR_t = G_t \cdot \sum_{a=15}^{49} \left[\left(\frac{h_t^p(a)}{G_t} \right) \cdot LFR_{t-(a+1)} \right] = G_t \cdot \overline{LFR_t} \tag{2.7}$$

where G_t is a period (year *t*) adjustment factor, while $\overline{LFR_t} = \sum_{a=15}^{49} \left[\left(\frac{h_t^p(a)}{G_t} \right) \cdot LFR_{t-(a+1)} \right]$ is a

[2] This is probably the origin of the term – total fertility rate.

[3] Many researchers call the fertility level factor the quantum component and the fertility timing factor the tempo component.

weighted average of the concerned (cohort) lifetime fertility rates. Equation (2.7) shows that, under the assumption of no mortality before the end of the reproductive lifespan, the (period) TFR in year t (as defined in equation (2.5)) can be decomposed into two components, i.e. a level (quantum) factor (i.e. $\overline{LFR_t}$) and a timing (tempo) factor (i.e. G_t).

Mathematically, equation (2.7) provides a general expression for the quantitative relationship between the (period) total fertility rate and the corresponding (cohort) lifetime fertility rates. Butz and Ward (1979) noticed the relationship expressed in equation (2.7) and called the quantity G_t the timing index (TI), and the quantity $\overline{LFR_t}$ the average completed fertility (AC).

Equation (2.7) also provides a way for decomposing the change in the (period) total fertility rate into different factors as follows:

$$TFR_{t+1} - TFR_t = G_{t+1} \cdot \overline{LFR_{t+1}} - G_t \cdot \overline{LFR_t} = E_{\text{level}} + E_{\text{timing}} + I \tag{2.8}$$

where $E_{\text{level}} = G_t \cdot (\overline{LFR_{t+1}} - \overline{LFR_t})$ is the net effect of the change in the fertility quantum component (i.e. assuming that the fertility tempo component remains unchanged from year t to year t+1); $E_{\text{timing}} = (G_{t+1} - G_t) \cdot \overline{LFR_t}$ is the net effect of the change in the fertility tempo component (i.e. assuming that the fertility quantum component remains unchanged from year t to year t+1); and $I = (G_{t+1} - G_t) \cdot (\overline{LFR_{t+1}} - \overline{LFR_t})$ is an interaction term, which reflects the joint effect of the simultaneous changes in both the fertility tempo and the fertility quantum components.

By its definition, the period quantity G_t is affected by the childbearing behaviors of the concerned birth cohorts (i.e. t-50, t-49, …, t-16) in year t. Theoretically, G_t can take numerical values between 0 and 35. If all women of all the birth cohorts (i.e. t-50, t-49, …, t-16) do not give any births in year t, then $G_t = 0$ (because in this case, we have $h^c_{t-16}(15) = h^c_{t-17}(16) = \cdots = h^c_{t-50}(49) = 0$). If all women of all the birth cohorts (i.e. t-50, t-49, …, t-16) deliver all their (lifetime) births in year t, then $G_t = 35$ (because in this case, we have $h^c_{t-16}(15) = h^c_{t-17}(16) = \cdots = h^c_{t-50}(49) = 1$). Obviously, the above situations are two extremes. In reality, the numerical values of G_t usually fall between 0.5 and 1.5. It is also obvious that if all the birth cohorts of women (i.e. t-50, t-49, …, t-16) follow the same standardized cohort age pattern of fertility, then $G_t = 1$.

Let $H_t^p(a) = h_t^p(a) / G_t$, $a = 15, 16, \ldots, 49$, then it is obvious that $H_t^p(a) \geq 0$ and $\sum_{a=15}^{49} H_t^p(a) = 1$. Therefore, $\{H_t^p(a) \mid a = 15, 16, \ldots, 49\}$ constitutes a standardized period (year *t*) age pattern (schedule). Thus, equation (2.7) can be rewritten as

$$TFR_t = G_t \cdot \sum_{a=15}^{49} [H_t^p(a) \cdot LFR_{t-(a+1)}] \tag{2.9}$$

Suppose that LFR_y can be expressed by the following n^{th}-degree polynomial of *y*:

$$LFR_y = \lambda_0 + \lambda_1 \cdot y + \lambda_2 \cdot y^2 + \cdots + \lambda_n \cdot y^n = \sum_{i=0}^{n} (\lambda_i \cdot y^i) \tag{2.10}$$

where n is a non-negative integer and λ_i, $i = 0, 1, 2, \ldots, n$, are the polynomial coefficients. Let $T = t - 1$, then from equations (2.9) and (2.10), we have

$$TFR_t = G_t \cdot \sum_{a=15}^{49} \left[H_t^p(a) \cdot \sum_{i=0}^{n} [\lambda_i \cdot (T-a)^i] \right] = G_t \cdot \sum_{i=0}^{n} \left[\lambda_i \cdot \sum_{a=15}^{49} [(T-a)^i \cdot H_t^p(a)] \right] \tag{2.11}$$

Let $Q_n = \sum_{i=0}^{n} \left[\lambda_i \cdot \sum_{a=15}^{49} [(T-a)^i \cdot H_t^p(a)] \right]$, then equation (2.11) becomes

$$TFR_t = G_t \cdot Q_n \tag{2.12}$$

In other words, under the polynomial assumption about LFR_y, we have $\overline{LFR_t} = Q_n$.

By the binomial theorem, we have

$$(T-a)^i = \sum_{j=0}^{i} \left[\left(\frac{i!}{j!(i-j)!} \right) \cdot T^{i-j} \cdot (-a)^j \right] = \sum_{j=0}^{i} \left[(-1)^j \cdot \left(\frac{i!}{j!(i-j)!} \right) \cdot T^{i-j} \cdot a^j \right] \tag{2.13}$$

Therefore, we obtain

$$\begin{aligned} Q_n &= \sum_{i=0}^{n} \left\{ \lambda_i \cdot \sum_{a=15}^{49} [(T-a)^i \cdot H_t^p(a)] \right\} \\ &= \sum_{i=0}^{n} \left\{ \lambda_i \cdot \sum_{a=15}^{49} \left\{ \sum_{j=0}^{i} \left[(-1)^j \cdot \left(\frac{i!}{j!(i-j)!} \right) \cdot T^{i-j} \cdot a^j \cdot H_t^p(a) \right] \right\} \right\} \\ &= \sum_{i=0}^{n} \left\{ \lambda_i \cdot \sum_{j=0}^{i} \left[(-1)^j \cdot \left(\frac{i!}{j!(i-j)!} \right) \cdot T^{i-j} \cdot \sum_{a=15}^{49} [a^j \cdot H_t^p(a)] \right] \right\} \end{aligned} \tag{2.14}$$

To further explore the period quantity Q_n, we need to examine the period curve $H_t^p(a)$. In this connection, the (statistical) moments are important measures for describing $H_t^p(a)$. There are two types of moments for describing a probability distribution, i.e. the absolute moments (about zero or origin) and the central moments (about the mean). Suppose that function $p(x)$ represents a probability distribution (i.e. $p(x)$ satisfies $p(x) \geq 0$ and $\sum_{all\ x} p(x) = 1$), then its moments are defined as follows:

The r^{th} absolute moment (about zero or origin) of $p(x)$ is defined as

$$\hat{M}_r\langle p \rangle = \sum_{all\ x} [x^r \cdot p(x)] \tag{2.15}$$

where r is a non-negative integer. It is obvious that $\hat{M}_0\langle p \rangle = 1$ and $\hat{M}_1\langle p \rangle = \mu\langle p \rangle$, where $\mu\langle p \rangle = \sum_{all\ x} [x \cdot p(x)]$ is the mean of $p(x)$.

The r^{th} central moment (about the mean) of $p(x)$ is defined as

$$\tilde{M}_r\langle p \rangle = \sum_{all\ x} [(x - \mu\langle p \rangle)^r \cdot p(x)] \tag{2.16}$$

where r is a non-negative integer. It is obvious that $\tilde{M}_0\langle p \rangle = 1$, $\tilde{M}_1\langle p \rangle = 0$, and $\tilde{M}_2\langle p \rangle = v\langle p \rangle$, where $v\langle p \rangle = \sum_{all\ x} [(x - \mu\langle p \rangle)^2 \cdot p(x)]$ is the variance of $p(x)$.

Other relevant properties of the moments are given in Annex A.

From equation (2.14), we have

$$Q_n = \sum_{i=0}^{n} \left\{ \lambda_i \cdot \sum_{j=0}^{i} \left[(-1)^j \cdot \left(\frac{i!}{j! \cdot (i-j)!} \right) \cdot T^{i-j} \cdot \hat{M}_j \langle H_t^p \rangle \right] \right\} \tag{2.17}$$

It is obvious from equation (2.17) that

$$\left. \begin{aligned} & Q_0 = \lambda_0 \cdot \hat{M}_0 \langle H_t^p \rangle = \lambda_0 \\ & Q_d = Q_{d-1} + \lambda_d \cdot \sum_{j=0}^{d} \left[(-1)^j \cdot \left(\frac{d!}{j! \cdot (d-j)!} \right) \cdot T^{d-j} \cdot \hat{M}_j \langle H_t^p \rangle \right], \quad d = 1, 2, \ldots, n \end{aligned} \right\} \tag{2.18}$$

Once the polynomial coefficients λ_i, $i = 0, 1, 2, \ldots, n$, are known, equation (2.17) shows that Q_n is a linear function of the absolute moments of $H_t^p(a)$, i.e. Q_n can be written in the following form

$$Q_n = \lambda_0 + \sum_{i=1}^{n} (\tilde{\lambda}_i \cdot \hat{M}_i \langle H_t^p \rangle) \tag{2.19}$$

Equation (2.19) shows that Q_n is determined by the polynomial coefficients and the absolute moments of the period curve $H_t^p(a)$.

Based on the general relationship, expressed in equation (2.12), we will explore some specific relationships between the (period) total fertility rate and the (cohort) lifetime fertility rates. In this regard, we will look at the period level component (i.e. Q_n) and the period timing component (i.e. G_t) separately, as the two components may be considered "independent of each other" from a mathematical point of view.

3. Some specific expressions of Q_n

3.1 The (cohort) lifetime fertility rate remains constant over time (birth cohort)

Under this assumption, we have $LFR_{t-16} = LFR_{t-17} = \cdots = LFR_{t-50}$, denoted as LFR. This is equivalent to taking $n = 0$ in equation (2.10). Therefore, $LFR_y = LFR = \lambda_0$. Then from equation (2.18), we have $Q_0 = \lambda_0 = LFR$. In this case, we have

$$TFR_t = G_t \cdot Q_0 = G_t \cdot LFR \tag{3.1}$$

Equation (3.1) shows that even if the level of cohort fertility (i.e. lifetime fertility rate) is invariant over time (cohort), the (period) total fertility rate may be greater than, equal to, or smaller than the (cohort) lifetime fertility rate depending on the period adjustment factor for year *t* (i.e. G_t). If $G_t = 1$ (women procreate in year *t* "normally"), then we have $TFR_t = LFR$. If $G_t > 1$ (women "favor" year *t* in terms of childbearing), then we have $TFR_t > LFR$. If $G_t < 1$ (women "avoid" year *t* in terms of childbearing), then we have $TFR_t < LFR$.

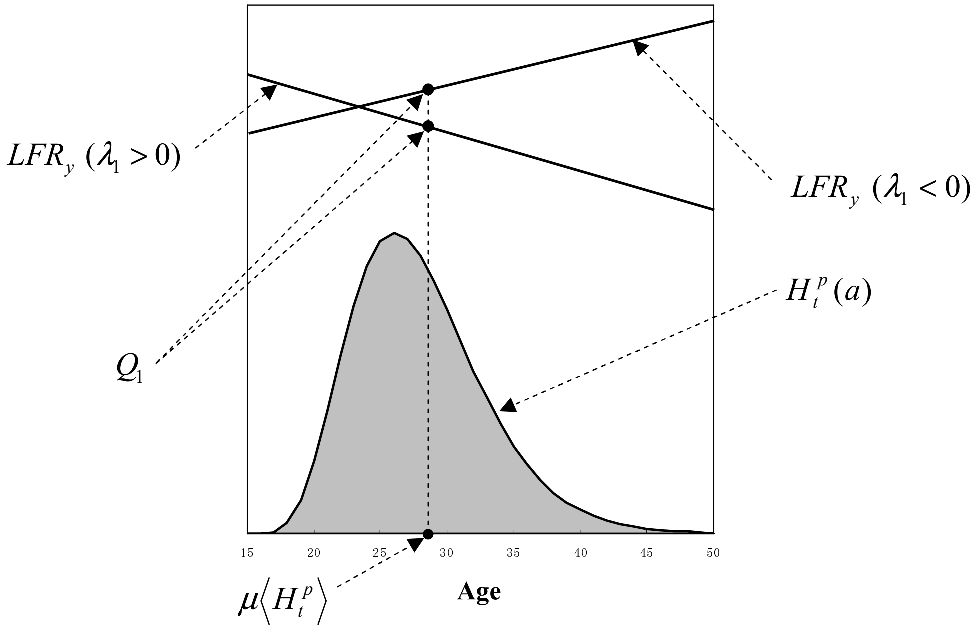
3.2 The (cohort) lifetime fertility rate changes linearly with time (birth cohort)

This is equivalent to taking $n = 1$ in equation (2.10), i.e. $LFR_y = \lambda_0 + \lambda_1 \cdot y$. Then from equation (2.18), we have

$$Q_1 = Q_0 + \lambda_1 \cdot (T - \hat{M}_1\langle H_t^p \rangle) = \lambda_0 + \lambda_1 \cdot (T - \hat{M}_1\langle H_t^p \rangle) = LFR_{T-\hat{M}_1\langle H_t^p \rangle} = LFR_{t-(\mu\langle H_t^p \rangle+1)} \quad (3.2)$$

In equation (3.2), $LFR_{t-(\mu\langle H_t^p \rangle+1)}$ is the lifetime fertility rate of birth cohort $t-(\mu\langle H_t^p \rangle+1)$, which is aged $\mu\langle H_t^p \rangle$ at the beginning of year *t*. Figure 3 shows the relationship between Q_1 and $\mu\langle H_t^p \rangle$.

Figure 3. Relationship between Q_1 and $\mu\langle H_t^p \rangle$.



Equation (3.2) shows that under the linear assumption, Q_1 is affected by the mean of the period curve $H_t^p(a)$, but not affected by its shape (e.g. variance, skewness, kurtosis).

In this case, we have

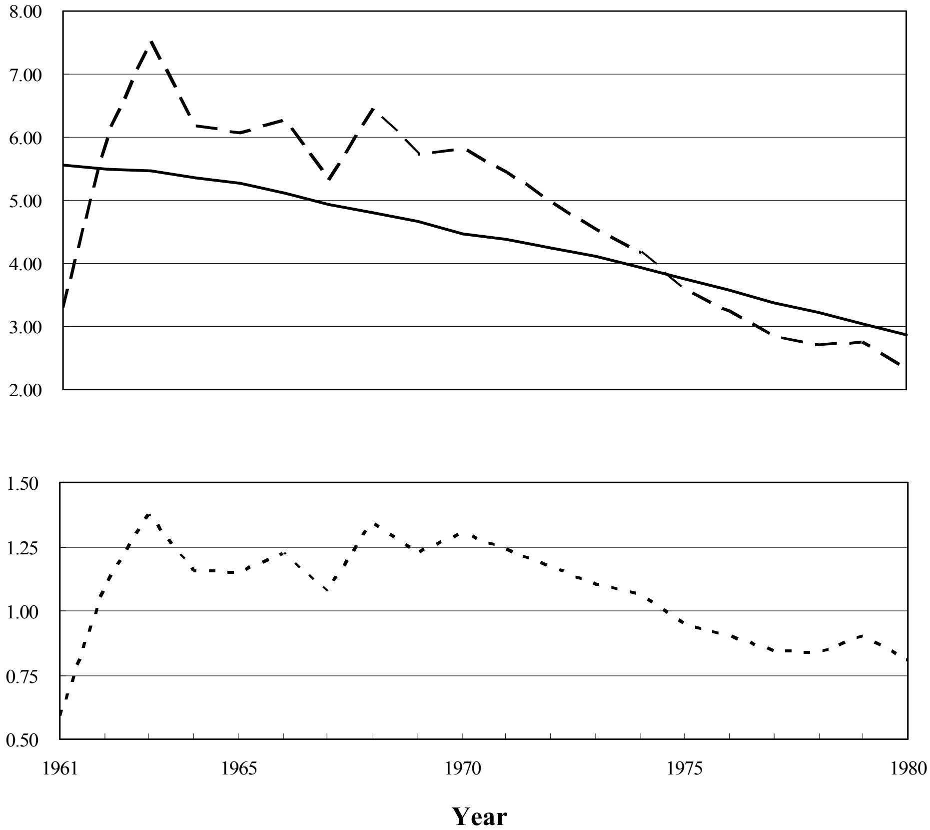
$$TFR_t = G_t \cdot Q_1 = G_t \cdot LFR_{t-(\mu\langle H_t^p \rangle+1)} \quad (3.3)$$

Equation (3.3) indicates that, under the assumption stated above, the mean age of the standardized period fertility curve $H_t^p(a)$ (i.e. $\mu\langle H_t^p \rangle$) plays a key role in determining the total fertility rate for year *t* (i.e. TFR_t).

Based on the data from China's 2‰ fertility survey conducted in 1988, we produced Figure 4, which shows that the (cohort) lifetime fertility rates (LFR_y) of Chinese women born during 1931-1950 declined almost linearly with time (cohort), with the coefficient of determination being $R^2 = 0.99$.

In producing Figure 4, $\mu\langle H_t^p \rangle$ was set at a constant of 29 years of age.

Figure 4. The values of LFR_y, TFR_t and G_t - China



3.3 The (cohort) lifetime fertility rate changes quadratically with time (birth cohort)

This is equivalent to taking $n = 2$ in equation (2.10), i.e. $LFR_y = \lambda_0 + \lambda_1 \cdot y + \lambda_2 \cdot y^2$. Therefore from equation (2.18), we have

$$
\begin{aligned}
Q_2 &= Q_1 + \lambda_2 \cdot [T^2 - 2 \cdot T \cdot \hat{M}_1\langle H_t^p \rangle + \hat{M}_2\langle H_t^p \rangle] \\
&= Q_1 + \lambda_2 \cdot [(T - \hat{M}_1\langle H_t^p \rangle)^2 + \hat{M}_2\langle H_t^p \rangle - (\hat{M}_1\langle H_t^p \rangle)^2] \\
&= \lambda_0 + \lambda_1 \cdot (T - \hat{M}_1\langle H_t^p \rangle) + \lambda_2 \cdot (T - \hat{M}_1\langle H_t^p \rangle)^2 \\
&\quad + \lambda_2 \cdot [\hat{M}_2\langle H_t^p \rangle - (\hat{M}_1\langle H_t^p \rangle)^2] \\
&= LFR_{T - \hat{M}_1\langle H_t^p \rangle} + \lambda_2 \cdot \hat{D}_2\langle H_t^p \rangle \\
&= LFR_{t - (\mu\langle H_t^p \rangle + 1)} + \lambda_2 \cdot (\sigma\langle H_t^p \rangle)^2 \qquad (3.4)
\end{aligned}
$$

In equation (3.4), $LFR_{t-(\mu\langle H_t^p \rangle+1)}$ is the lifetime fertility rate of birth cohort $t - (\mu\langle H_t^p \rangle + 1)$ (which is aged $\mu\langle H_t^p \rangle$ at the beginning of year *t*), and $\lambda_2 \cdot (\sigma\langle H_t^p \rangle)^2$ is a modification term. It is obvious that (i) when $\lambda_2 > 0$ (i.e. the parabola opens upwards), the modification term is positive, and (ii) when $\lambda_2 < 0$ (i.e. the parabola opens downwards), the modification term is negative. Equation (3.4) also shows that under the quadratic assumption, Q_2 is not only affected by the mean of the period curve $H_t^p(a)$, but also affected by its standard deviation. In this case, we have

$$TFR_t = G_t \cdot Q_2 = G_t \cdot \left[LFR_{t-(\mu\langle H_t^p \rangle+1)} + \lambda_2 \cdot (\sigma\langle H_t^p \rangle)^2 \right] \tag{3.5}$$

3.4 The (cohort) lifetime fertility rate changes cubically with time (birth cohort)

This is equivalent to taking $n = 3$ in equation (2.10), i.e. $LFR_y = \lambda_0 + \lambda_1 \cdot y + \lambda_2 \cdot y^2 + \lambda_3 \cdot y^3$. Therefore from equation (2.18), we have

$$\begin{aligned} Q_3 &= Q_2 + \lambda_3 \cdot [T^3 - 3 \cdot T^2 \cdot \hat{M}_1\langle H_t^p \rangle + 3 \cdot T \cdot \hat{M}_2\langle H_t^p \rangle - \hat{M}_3\langle H_t^p \rangle] \\ &= \lambda_0 + \lambda_1 \cdot (T - \hat{M}_1\langle H_t^p \rangle) + \lambda_2 \cdot (T - \hat{M}_1\langle H_t^p \rangle)^2 + \lambda_3 \cdot (T - \hat{M}_1\langle H_t^p \rangle)^3 \\ &\quad + \lambda_2 \cdot \hat{D}_2\langle H_t^p \rangle \\ &\quad + \lambda_3 \cdot (3 \cdot T \cdot \hat{D}_2\langle H_t^p \rangle - \hat{D}_3\langle H_t^p \rangle) \\ &= LFR_{t-(\mu\langle H_t^p \rangle+1)} + \lambda_2 \cdot \hat{D}_2\langle H_t^p \rangle + \lambda_3 \cdot (3 \cdot T \cdot \hat{D}_2\langle H_t^p \rangle - \hat{D}_3\langle H_t^p \rangle) \end{aligned} \tag{3.6}$$

Since

$$3 \cdot T \cdot \hat{D}_2\langle H_t^p \rangle - \hat{D}_3\langle H_t^p \rangle = (\sigma\langle H_t^p \rangle)^2 \cdot \omega_3$$

where $\omega_3 = 3 \cdot T - 3 \cdot \mu\langle H_t^p \rangle - \sigma\langle H_t^p \rangle \cdot s\langle H_t^p \rangle$, equation (3.6) can be rewritten as

$$\begin{aligned} Q_3 &= LFR_{t-(\mu\langle H_t^p \rangle+1)} + \lambda_2 \cdot (\sigma\langle H_t^p \rangle)^2 + \lambda_3 \cdot (\sigma\langle H_t^p \rangle)^2 \cdot \omega_3 \\ &= LFR_{t-(\mu\langle H_t^p \rangle+1)} + (\lambda_2 + \lambda_3 \cdot \omega_3) \cdot (\sigma\langle H_t^p \rangle)^2 \end{aligned} \tag{3.7}$$

In this case, we have

$$TFR_t = G_t \cdot Q_3 = G_t \cdot \left[LFR_{t-(\mu\langle H_t^p \rangle+1)} + (\lambda_2 + \lambda_3 \cdot \omega_3) \cdot (\sigma\langle H_t^p \rangle)^2 \right] \tag{3.8}$$

3.5 The (cohort) lifetime fertility rate changes quartically with time (birth cohort)

This is equivalent to taking $n = 4$ in equation (2.10), i.e.

$LFR_y = \lambda_0 + \lambda_1 \cdot y + \lambda_2 \cdot y^2 + \lambda_3 \cdot y^3 + \lambda_4 \cdot y^4$. Therefore from equation (2.18), we have

$$\begin{aligned}
Q_4 &= Q_3 + \lambda_4 \cdot [T^4 - 4 \cdot T^3 \cdot \hat{M}_1\langle H_t^p \rangle + 6 \cdot T^2 \cdot \hat{M}_2\langle H_t^p \rangle - 4 \cdot T \cdot \hat{M}_3\langle H_t^p \rangle + \hat{M}_4\langle H_t^p \rangle] \\
&= \lambda_0 + \lambda_1 \cdot (T - \hat{M}_1\langle H_t^p \rangle) + \lambda_2 \cdot (T - \hat{M}_1\langle H_t^p \rangle)^2 + \lambda_3 \cdot (T - \hat{M}_1\langle H_t^p \rangle)^3 + \lambda_4 \cdot (T - \hat{M}_1\langle H_t^p \rangle)^4 \\
&\quad + \lambda_2 \cdot \hat{D}_2\langle H_t^p \rangle \\
&\quad + \lambda_3 \cdot (3 \cdot T \cdot \hat{D}_2\langle H_t^p \rangle - \hat{D}_3\langle H_t^p \rangle) \\
&\quad + \lambda_4 \cdot (6 \cdot T^2 \cdot \hat{D}_2\langle H_t^p \rangle - 4 \cdot T \cdot \hat{D}_3\langle H_t^p \rangle + \hat{D}_4\langle H_t^p \rangle) \\
&= LFR_{t-(\mu\langle H_t^p \rangle+1)} + \lambda_2 \cdot (\sigma\langle H_t^p \rangle)^2 + \lambda_3 \cdot [3 \cdot T \cdot \hat{D}_2\langle H_t^p \rangle - \hat{D}_3\langle H_t^p \rangle] \\
&\quad + \lambda_4 \cdot [6 \cdot T^2 \cdot \hat{D}_2\langle H_t^p \rangle - 4 \cdot T \cdot \hat{D}_3\langle H_t^p \rangle + \hat{D}_4\langle H_t^p \rangle] \qquad (3.9)
\end{aligned}$$

Since

$$6 \cdot T^2 \cdot \hat{D}_2\langle H_t^p \rangle - 4 \cdot T \cdot \hat{D}_3\langle H_t^p \rangle + \hat{D}_4\langle H_t^p \rangle = (\sigma\langle H_t^p \rangle)^2 \cdot \omega_4 \qquad (3.10)$$

where $\omega_4 = 6 \cdot T^2 - 6 \cdot \mu\langle H_t^p \rangle \cdot (2 \cdot T - 1) - 4 \cdot \sigma\langle H_t^p \rangle \cdot s\langle H_t^p \rangle \cdot (T - \mu\langle H_t^p \rangle) + (\sigma\langle H_t^p \rangle)^2 \cdot k\langle H_t^p \rangle$, equation (3.9) can be rewritten as

$$\begin{aligned}
Q_4 &= LFR_{t-(\mu\langle H_t^p \rangle+1)} + \lambda_2 \cdot (\sigma\langle H_t^p \rangle)^2 + \lambda_3 \cdot \omega_3 \cdot (\sigma\langle H_t^p \rangle)^2 + \lambda_4 \cdot \omega_4 \cdot (\sigma\langle H_t^p \rangle)^2 \\
&= LFR_{t-(\mu\langle H_t^p \rangle+1)} + (\lambda_2 + \lambda_3 \cdot \omega_3 + \lambda_4 \cdot \omega_4) \cdot (\sigma\langle H_t^p \rangle)^2 \qquad (3.11)
\end{aligned}$$

In this case, we have

$$TFR_t = G_t \cdot Q_4 = G_t \cdot \left[LFR_{t-(\mu\langle H_t^p \rangle+1)} + (\lambda_2 + \lambda_3 \cdot \omega_3 + \lambda_4 \cdot \omega_4) \cdot (\sigma\langle H_t^p \rangle)^2 \right] \qquad (3.12)$$

4. Specific expressions of G_t - Assumption I

From the discussions above, we have noticed that the period quantity $G_t = \sum_{a=15}^{49} h_t^p(a)$ (for year *t*) is a very important factor in terms of linking the (period) total fertility rate to the corresponding (cohort) lifetime fertility rates.

Following a similar approach of Ryder (1964), we assume that for each age *a*, the time sequence $\{ h_y^c(a) \mid y = t-50, t-49, \ldots, t-16 \}$ can be represented by the following m^{th}-degree polynomial of *y*:

$$h_y^c(a) = \beta_0(a) + \beta_1(a) \cdot y + \beta_2(a) \cdot y^2 + \cdots + \beta_m(a) \cdot y^m = \sum_{i=0}^{m}[\beta_i(a) \cdot y^i] \tag{4.1}$$

where m is a positive integer and $\beta_i(a)$, $i = 0, 1, 2, \ldots, m$, are the polynomial coefficients. Since for each birth cohort y, we have $\sum_{a=15}^{49} h_y^c(a) = 1$, it follows that

$$\sum_{a=15}^{49}\sum_{i=0}^{m}[\beta_i(a) \cdot y^i] = \sum_{i=0}^{m}\left[\left(\sum_{a=15}^{49}\beta_i(a)\right) \cdot y^i\right] = 1 \tag{4.2}$$

Let $\pi_i = \sum_{a=15}^{49}\beta_i(a)$, then we have

$$\pi_0 + \pi_1 \cdot y + \pi_2 \cdot y^2 + \cdots + \pi_m \cdot y^m = \sum_{i=0}^{m}(\pi_i \cdot y^i) = 1 \tag{4.3}$$

We define an m^{th}-degree polynomial of y as follows: $q_m(y) = \sum_{i=0}^{m}(\pi_i \cdot y^i) - 1$, where $\pi_m \neq 0$, then it is obvious that the polynomial $q_m(y)$ has 35 real (integer) roots, i.e. $y = t$-50, t-49, …, t-16. It can be proved that when $m < 35$, there must be $\pi_0 = 1$ and $\pi_i = 0$ ($i = 1, 2, \ldots, m$). (Proof by contradiction: If $\pi_m \neq 0$, then according to the fundamental theorem of algebra, the polynomial $q_m(y)$ has at most m real roots, which contradicts the fact that the polynomial $q_m(y)$ has 35 real roots). Therefore, we have $\pi_m = 0$. Following the same logic, we have $\pi_{m-1} = 0$, …, $\pi_1 = 0$. Hence, from equation (A.3), we have $\pi_0 = 1$.

From the definition of G_t and letting $T = t - 1$, we have

$$G_t = \sum_{a=15}^{49} h_t^p(a) = \sum_{a=15}^{49} h_{t-(a+1)}^c(a) = \sum_{a=15}^{49}\sum_{i=0}^{m}[\beta_i(a) \cdot (T-a)^i] \tag{4.4}$$

Then by applying the binomial theorem to equation (4.4), we have

$$G_t = \sum_{a=15}^{49}\sum_{i=0}^{m}\left\{\beta_i(a) \cdot \sum_{j=0}^{i}\left[(-1)^j \cdot \left(\frac{i!}{j! \cdot (i-j)!}\right) \cdot T^{i-j} \cdot a^j\right]\right\}$$

$$= \sum_{i=0}^{m}\sum_{j=0}^{i}\left\{\left[(-1)^j \cdot \left(\frac{i!}{j! \cdot (i-j)!}\right) \cdot T^{i-j}\right] \cdot \sum_{a=15}^{49}[a^j \cdot \beta_i(a)]\right\} \tag{4.5}$$

Equation (4.5) provides a general expression for G_t, under assumption expressed in equation (4.1).

For each birth cohort *y*, we denote the mean age of the standardized cohort fertility schedule $h_y^c(a)$ as $\mu\langle h_y^c\rangle$, $y = t-50, t-49, \ldots, t-16$, then we have

$$\mu\langle h_y^c\rangle = \sum_{a=15}^{49}[a \cdot h_y^c(a)] = \sum_{a=15}^{49}\left[a \cdot \sum_{i=0}^{m}[\beta_i(a) \cdot y^i]\right] = \sum_{i=0}^{m}\left[\left(\sum_{a=15}^{49}[a \cdot \beta_i(a)]\right) \cdot y^i\right] \quad (4.6)$$

Equation (4.6) shows that under the assumption stated in equation (4.1), $\mu\langle h_y^c\rangle$ is also a polynomial of degree *m*.

For each birth cohort *y*, we denote the variance of the cohort fertility curve $h_y^c(a)$ as $v\langle h_y^c\rangle$, then we have $v\langle h_y^c\rangle = \sum_{a=15}^{49}[(a - \mu\langle h_y^c\rangle)^2 \cdot h_y^c(a)]$, *y* = *t*-50, *t*-49, …, *t*-16, and

$$v\langle h_y^c\rangle = \sum_{a=15}^{49}[a^2 \cdot h_y^c(a)] - (\mu\langle h_y^c\rangle)^2 = \sum_{a=15}^{49}\left[a^2 \cdot \left(\sum_{i=0}^{m}[\beta_i(a) \cdot y^i]\right)\right] - (\mu\langle h_y^c\rangle)^2$$

$$= \sum_{i=0}^{m}\left[\left(\sum_{a=15}^{49}[a^2 \cdot \beta_i(a)]\right) \cdot y^i\right] - (\mu\langle h_y^c\rangle)^2 \quad (4.7)$$

Equation (4.7) shows that under the assumption stated in equation (4.1), $v\langle h_y^c\rangle$ is a polynomial of degree $2 \cdot m$.

Now, let's consider two specific cases.

4.1 For each age a $(a = 15, 16, \ldots, 49)$*, the time sequence* $\{h_y^c(a) \mid y = t-50, t-49, \ldots, t-16\}$ *can be represented by a linear function of y*

This is equivalent to taking $m = 1$ in equation (4.1), i.e. $h_y^c(a) = \beta_0(a) + \beta_1(a) \cdot y$, where $\beta_0(a)$ is the intercept and $\beta_1(a)$ is the slope of the straight line. Then, from the discussion above, we know that $\pi_0 = \sum_{a=15}^{49}\beta_0(a) = 1$ and $\pi_1 = \sum_{a=15}^{49}\beta_1(a) = 0$. Therefore, from equation (4.5), we have

$$G_t = \sum_{a=15}^{49}\beta_0(a) + T \cdot \sum_{a=15}^{49}\beta_1(a) - \sum_{a=15}^{49}[a \cdot \beta_1(a)] = 1 - \sum_{a=15}^{49}[a \cdot \beta_1(a)] \quad (4.8)$$

In the mean time, from equation (4.6), we have

$$\mu\langle h_y^c\rangle = \sum_{a=15}^{49}[a\cdot\beta_0(a)] + \left(\sum_{a=15}^{49}[a\cdot\beta_1(a)]\right)\cdot y \tag{4.9}$$

Equation (4.9) shows that under the assumption stated, the mean age of the standardized cohort fertility schedule $h_y^c(a)$, $y = t-50, t-49, \dots, t-16$, is also a linear function of the birth cohort (y), with the intercept being $\sum_{a=15}^{49}[a\cdot\beta_0(a)]$ and the slope $\sum_{a=15}^{49}[a\cdot\beta_1(a)]$. From equation (4.9), we also have $\sum_{a=15}^{49}[a\cdot\beta_1(a)] = \frac{1}{34}\cdot(\mu\langle h_{t-16}^c\rangle - \mu\langle h_{t-50}^c\rangle)$.

Let $\varphi = \sum_{a=15}^{49}[a\cdot\beta_1(a)]$, then equation (4.8) becomes $G_t = 1-\varphi$, where φ is the slope of $\mu\langle h_y^c\rangle$. In other words, under the assumption stated, G_t is equal to one minus the slope (rate of change) of the mean age of the cohort fertility curve $h_y^c(a)$, y = t-50, t-49, …, t-16.

Taking the first derivative with respect to y on both sides of equation (4.9), we obtain

$$[\mu\langle h_y^c\rangle]'_y = \sum_{a=15}^{49}[a\cdot\beta_1(a)] = \varphi \tag{4.10}$$

Therefore, G_t can also be written as $G_t = 1-[\mu\langle h_y^c\rangle]'_y$.

Under different assumptions, Ryder (1964) obtained a similar result by using the moment approach. It is obvious from equation (4.10) that (i) if the mean ages of the standardized cohort fertility curves $h_y^c(a)$ increase from cohort to cohort (i.e. women postpone childbearing), then $[\mu\langle h_y^c\rangle]'_y > 0$ and therefore $G_t < 1$; and (ii) if the mean ages of the standardized cohort fertility curves $h_y^c(a)$ decrease from cohort to cohort (i.e. women advance childbearing), then $[\mu\langle h_y^c\rangle]'_y > 0$ and therefore $G_t > 1$.

From equation (4.7), we have the variance of the cohort fertility curve $h_y^c(a)$ (y = t-50, t-49, …, t-16) as follows

$$v\langle h_y^c\rangle = \sum_{a=15}^{49}[a^2\cdot\beta_0(a)] + \left(\sum_{a=15}^{49}[a^2\cdot\beta_1(a)]\right)\cdot y - (\mu\langle h_y^c\rangle)^2 \tag{4.11}$$

Equation (4.11) shows that under the assumption stated, the variance of the cohort fertility curve

$h_y^c(a)$ (y = t-50, t-49, …, t-16) is a quadratic function of the birth cohort (y), with the coefficient of the quadratic term (i.e. y^2) being $-\left(\sum_{a=15}^{49}[a \cdot \beta_1(a)]\right)^2$ (<0). Therefore, $v\langle h_y^c \rangle$ (y = t-50, t-49, …, t-16) is a parabola, which opens downwards.

$$\mu\langle H_t^p \rangle = \sum_{a=15}^{49}[a \cdot H_t^p(a)] = \frac{1}{G_t} \cdot \sum_{a=15}^{49}[a \cdot h_t^p(a)] = \frac{1}{G_t} \cdot \sum_{a=15}^{49}[a \cdot h_{t-(a+1)}^c(a)]$$

$$= \frac{1}{G_t} \cdot \sum_{a=15}^{49}\{\, a \cdot [\beta_0(a) + \beta_1(a) \cdot (t-a-1)]\}$$

$$= \frac{1}{G_t} \cdot \left\{\sum_{a=15}^{49}[a \cdot \beta_0(a)] + (t-1) \cdot \sum_{a=15}^{49}[a \cdot \beta_1(a)] - \sum_{a=15}^{49}[a^2 \cdot \beta_1(a)]\right\} \qquad (4.12)$$

Substituting equation (4.8) into equation (4.12), we have

$$\mu\langle H_t^p \rangle = \frac{\sum_{a=15}^{49}[a \cdot \beta_0(a)] + (t-1) \cdot \sum_{a=15}^{49}[a \cdot \beta_1(a)] - \sum_{a=15}^{49}[a^2 \cdot \beta_1(a)]}{1 - \sum_{a=15}^{49}[a \cdot \beta_1(a)]} \qquad (4.13)$$

Next, we will conduct a numerical simulation so that we can have a concrete understanding of the theoretical relationships discussed above. For this purpose, we will use the following Gamma function for the simulation.

$$g(a) = \begin{cases} K \cdot (a-a_0)^A \cdot e^{-B \cdot (a-a_0)}, & when \quad a \ge a_0 \\ 0, & when \quad a < a_0 \end{cases} \qquad (4.14)$$

The properties of the above-defined Gamma function are discussed in detail in Annex B.

Suppose that (i) the standardized cohort fertility schedule $\{h_{t-50}^c(a) \mid a = 15, 16, \ldots, 49\}$ (i.e. the oldest birth cohort) follows a Gamma function with a mean of 28 and a standard deviation of 5, and (ii) the standardized cohort fertility schedule $\{h_{t-16}^c(a) \mid a = 15, 16, \ldots, 49\}$ (i.e. the youngest birth cohort) follows a Gamma function with a mean of 32 and a standard deviation of 5. All the standardized cohort fertility schedules between the oldest and the youngest birth cohorts are then generated by linear interpolation age by age between the oldest and the youngest birth cohorts, i.e. for each age a (a = 15, 16, …, 49), the $h_y^c(a)$ is calculated as follows:

$$h_y^c(a) = h_{t-50}^c(a) + \Delta(a)[y-(t-50)], \quad y = t-49, t-48, \ldots, t-17 \tag{4.15}$$

where $\Delta(a) = [h_{t-16}^c(a) - h_{t-50}^c(a)]/34$. It can be easily proved that $h_y^c(a)$ generated as per equation (4.13) satisfies $h_y^c(a) \geq 0$ and $\sum_{a=15}^{49} h_y^c(a) = 1$.

Figure 5. Intercept by age (i.e. $\beta_0(a)$)

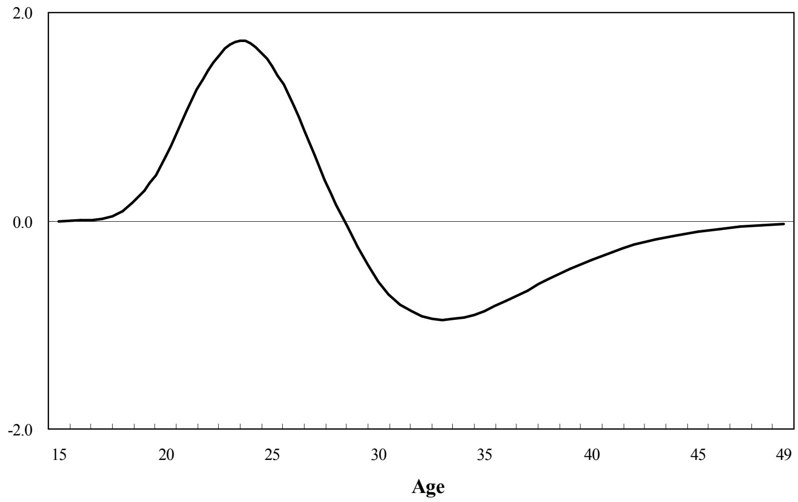


Figure 6. Slope by age (i.e. $\beta_1(a)$)

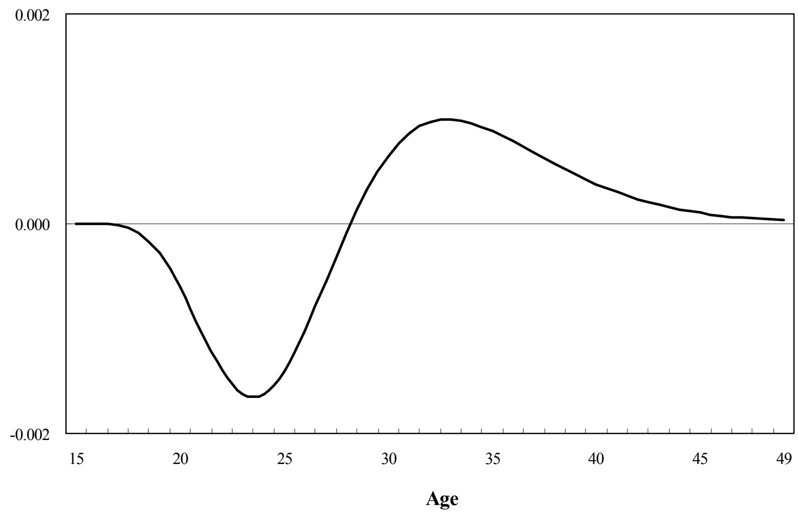
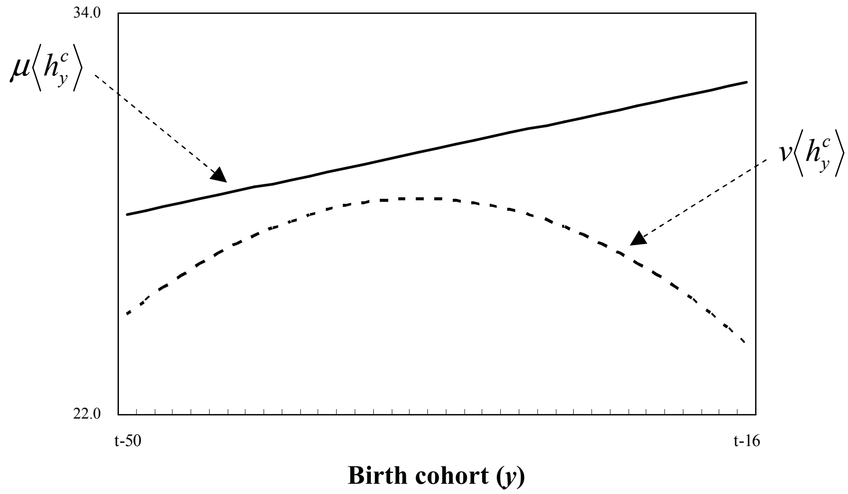


Figure 7. Mean ($\mu\langle h_y^c \rangle$) and variance ($v\langle h_y^c \rangle$) of $h_y^c(a)$



4.2 For each age a $(a = 15, 16, \ldots, 49)$*, the time sequence* $\left\{ h_y^c(a) \mid y = t-50, t-49, \ldots, t-16 \right\}$ *can be represented by a quadratic function of y*

This is equivalent to taking $m = 2$ in equation (4.1), i.e. $h_y^c(a) = \beta_0(a) + \beta_1(a) \cdot y + \beta_2(a) \cdot y^2$. Then, from the discussion above, we know that $\pi_0 = \sum_{a=15}^{49} \beta_0(a) = 1$, $\pi_1 = \sum_{a=15}^{49} \beta_1(a) = 0$ and $\pi_2 = \sum_{a=15}^{49} \beta_2(a) = 0$. Therefore, from equation (4.5), we have

$$G_t = \sum_{a=15}^{49} \beta_0(a) - \sum_{a=15}^{49}[a \cdot \beta_1(a)] - 2 \cdot T \cdot \sum_{a=15}^{49}[a \cdot \beta_2(a)] + \sum_{a=15}^{49}[a^2 \cdot \beta_2(a)]$$

$$= 1 - \sum_{a=15}^{49}[a \cdot \beta_1(a)] - 2 \cdot T \cdot \sum_{a=15}^{49}[a \cdot \beta_2(a)] + \sum_{a=15}^{49}[a^2 \cdot \beta_2(a)] \quad (4.16)$$

In this case, the mean age of the standardized cohort fertility curve $h_y^c(a)$ is

$$\mu\langle h_y^c \rangle = \sum_{a=15}^{49}[a \cdot \beta_0(a)] + \left(\sum_{a=15}^{49}[a \cdot \beta_1(a)] \right) \cdot y + \left(\sum_{a=15}^{49}[a \cdot \beta_2(a)] \right) \cdot y^2 \quad (4.17)$$

Equation (4.17) shows that under the assumption stated, the mean age of the cohort fertility curve $h_y^c(a)$ is also a quadratic function of the birth cohort (*y*). The variance of $h_y^c(a)$ is

$$v\langle h_y^c \rangle = \sum_{a=15}^{49}[(a - \mu\langle h_y^c \rangle)^2 \cdot h_y^c(a)] = \sum_{a=15}^{49}[a^2 \cdot h_y^c(a)] - (\mu\langle h_y^c \rangle)^2$$

$$= \sum_{a=15}^{49} [a^2 \cdot \beta_0(a)] + \left(\sum_{a=15}^{49} [a^2 \cdot \beta_1(a)] \right) \cdot y + \left(\sum_{a=15}^{49} [a^2 \cdot \beta_2(a)] \right) \cdot y^2 - (\mu\langle h_y^c \rangle)^2 \quad (4.18)$$

Equation (4.18) shows that under the assumption stated, the variance of the cohort fertility curve $h_y^c(a)$ is a quartic function of the birth cohort (y). From equation (4.16), we obtain

$$(\mu\langle h_y^c \rangle)^2 + v\langle h_y^c \rangle = \sum_{a=15}^{49} [a^2 \cdot \beta_0(a)] + \left(\sum_{a=15}^{49} [a^2 \cdot \beta_1(a)] \right) \cdot y + \left(\sum_{a=15}^{49} [a^2 \cdot \beta_2(a)] \right) \cdot y^2 \quad (4.19)$$

Taking the second derivative with respect to y on both sides of equations (4.17) and (4.19), we have

$$[\mu\langle h_y^c \rangle]''_y = 2 \cdot \sum_{a=15}^{49} [a \cdot \beta_2(a)] \quad (4.20)$$

$$[(\mu\langle h_y^c \rangle)^2 + v\langle h_y^c \rangle]''_y = 2 \cdot \sum_{a=15}^{49} [a^2 \cdot \beta_2(a)] \quad (4.21)$$

Incorporating equations (4.20) and (4.21) into equation (4.14), we get

$$G_t = 1 - \sum_{a=15}^{49} [a \cdot \beta_1(a)] - T \cdot [\mu\langle h_y^c \rangle]''_y + \frac{1}{2} \cdot [(\mu\langle h_y^c \rangle)^2]''_y + \frac{1}{2} \cdot [v\langle h_y^c \rangle]''_y \quad (4.22)$$

From equation (4.17), we have

$$\mu\langle h_{t-50}^c \rangle = \sum_{a=15}^{49} [a \cdot \beta_0(a)] + \left(\sum_{a=15}^{49} [a \cdot \beta_1(a)] \right) \cdot (T - 49) + \left(\sum_{a=15}^{49} [a \cdot \beta_2(a)] \right) \cdot (T - 49)^2 \quad (4.23)$$

$$\mu\langle h_{t-16}^c \rangle = \sum_{a=15}^{49} [a \cdot \beta_0(a)] + \left(\sum_{a=15}^{49} [a \cdot \beta_1(a)] \right) \cdot (T - 15) + \left(\sum_{a=15}^{49} [a \cdot \beta_2(a)] \right) \cdot (T - 15)^2 \quad (4.24)$$

where $T = t - 1$. Subtracting equation (4.23) from equation (4.24), we obtain

$$\begin{aligned} \mu\langle h_{t-16}^c \rangle - \mu\langle h_{t-50}^c \rangle &= 34 \cdot \sum_{a=15}^{49} [a \cdot \beta_1(a)] + (68 \cdot T - 2176) \cdot \sum_{a=15}^{49} [a \cdot \beta_2(a)] \\ &= 34 \cdot \sum_{a=15}^{49} [a \cdot \beta_1(a)] + \left(\frac{68 \cdot T - 2176}{2} \right) \cdot [\mu\langle h_y^c \rangle]''_y \\ &= 34 \cdot \sum_{a=15}^{49} [a \cdot \beta_1(a)] + (34 \cdot T - 1088) \cdot [\mu\langle h_y^c \rangle]''_y \end{aligned} \quad (4.25)$$

Therefore, we have

$$\sum_{a=15}^{49} [a \cdot \beta_1(a)] = \frac{1}{34} \left[(\mu\langle h_{t-16}^c \rangle - \mu\langle h_{t-50}^c \rangle) - (34 \cdot T - 1088) \cdot [\mu\langle h_y^c \rangle]''_y \right] \quad (4.26)$$

Finally, equation (4.22) becomes

$$G_t = 1 - \frac{1}{34}\Big[(\mu\langle h_{t-16}^c\rangle - \mu\langle h_{t-50}^c\rangle) - (34 \cdot T - 1088)\cdot[\mu\langle h_y^c\rangle]_y'' \Big]$$

$$- T\cdot[\mu\langle h_y^c\rangle]_y'' + \frac{1}{2}\cdot[(\mu\langle h_y^c\rangle)^2]_y'' + \frac{1}{2}\cdot[v\langle h_y^c\rangle]_y''$$

$$= 1 - \frac{1}{34}\Big[(\mu\langle h_{t-16}^c\rangle - \mu\langle h_{t-50}^c\rangle) - (35 \cdot T - 1088)\cdot[\mu\langle h_y^c\rangle]_y'' \Big]$$

$$+ \frac{1}{2}\cdot[(\mu\langle h_y^c\rangle)^2]_y'' + \frac{1}{2}\cdot[v\langle h_y^c\rangle]_y'' \qquad (4.27)$$

where $\mu\langle h_{t-16}^c\rangle - \mu\langle h_{t-50}^c\rangle$ can be regarded as the amount of "shift" between the two standardized cohort fertility schedules $h_{t-16}^c(a)$ and $h_{t-50}^c(a)$.

5. A specific expression of G_t - Assumption II

We assume that, for each birth cohort *y* (*y* = *t*-50, *t*-49, …, *t*-16), its standardized fertility schedule $h_y^c(a)$ is a continuous function of age *a*. Therefore, we have $h_y^c(a) \geq 0$ and $\int_{15}^{50} h_y^c(a)da = 1$. In addition, we designate the birth cohort *t*-50 (i.e. the birth cohort that reached the oldest childbearing age at the beginning of year *t*) as the benchmark cohort. For $h_y^c(a)$, we symbolize its mean, variance, skewness and kurtosis as follows:

$$\text{Mean: } \mu\langle h_y^c\rangle = \int_{15}^{50} [a \cdot h_y^c(a)]da \qquad (5.1)$$

$$\text{Variance: } v\langle h_y^c\rangle = \int_{15}^{50} [(a - \mu\langle h_y^c\rangle)^2 \cdot h_y^c(a)]da \qquad (5.2)$$

$$\text{Skewness: } s\langle h_y^c\rangle = \int_{15}^{50} \left[\left(\frac{a - \mu\langle h_y^c\rangle}{\sigma\langle h_y^c\rangle} \right)^3 \cdot h_y^c(a) \right] da \qquad (5.3)$$

$$\text{Kurtosis: } k\langle h_y^c\rangle = \int_{15}^{50} \left[\left(\frac{a - \mu\langle h_y^c\rangle}{\sigma\langle h_y^c\rangle} \right)^4 \cdot h_y^c(a) \right] da \qquad (5.4)$$

where $\sigma\langle h_y^c\rangle = \sqrt{v\langle h_y^c\rangle}$ represents the standard deviation of $h_y^c(a)$.

Now, we assume that each cohort curve $h_y^c(a)$, $y = t$-49, t-48, …, t-16, shifts along the age-axis by a constant amount δ (with no change in the shape of the curve, see Figure 8) relative to the curve of the preceeding birth cohort (i.e. y-1), i.e. $h_y^c(a) = h_{y-1}^c(a-\delta)$, $y = t$-49, t-48, …, t-16. Hence, we have $h_{t-(a+1)}^c(a) = h_{t-50}^c(a-(49-a)\cdot\delta)$. Therefore, from equation (5.1), we have

$$\mu\langle h_y^c\rangle = \int_{15}^{50}[a\cdot h_y^c(a)]da = \int_{15}^{50}[a\cdot h_{y-1}^c(a-\delta)]da \tag{5.5}$$

Let $u = a-\delta$, then $a = u+\delta$ and $da = du$. Therefore, from equation (5.5), we obtain

$$\mu\langle h_y^c\rangle = \int_{15}^{50}[(u+\delta)\cdot h_{y-1}^c(u)]du = \int_{15}^{50}[u\cdot h_{y-1}^c(u)]du + \delta\cdot\int_{15}^{50} h_{y-1}^c(u)du = \mu\langle h_{y-1}^c\rangle + \delta \tag{5.6}$$

Equation (5.6) shows that under the assumption stated, the mean age of $h_y^c(a)$ also shifts towards the same direction (when $\delta > 0$, the standardized cohort fertility curves shift towards the right side of the age-axis (i.e. higher ages); when $\delta < 0$, the standardized cohort fertility curves shift towards the left side of the age-axis (i.e. lower ages)) and by the same amount (i.e. $|\delta|$) as compared to $h_{y-1}^c(a)$. From equation (5.6), we have

$$\mu\langle h_y^c\rangle = \mu\langle h_{t-50}^c\rangle + [y-(t-50)]\cdot\delta,\quad y = t-50, t-49, \ldots, t-16 \tag{5.7}$$

Therefore, $\mu\langle h_y^c\rangle$ is a linear function of y. Taking the first derivative with respect to y on both sides, we obtain $[\mu\langle h_y^c\rangle]'_y = \delta$.

Similarly, it can be proved that $v\langle h_y^c\rangle = v\langle h_{y-1}^c\rangle$, $s\langle h_y^c\rangle = s\langle h_{y-1}^c\rangle$, and $k\langle h_y^c\rangle = k\langle h_{y-1}^c\rangle$.

From the discussions above, we have

$$G_t = \sum_{a=15}^{49} h_t^p(a) = \sum_{a=15}^{49} h_{t-(a+1)}^c(a) = \sum_{a=15}^{49} h_{t-50}^c(a-(49-a)\cdot\delta) = \sum_{a=15}^{49c} h_{t-50}^c((1+\delta)\cdot a - 49\cdot\delta) \tag{5.8}$$

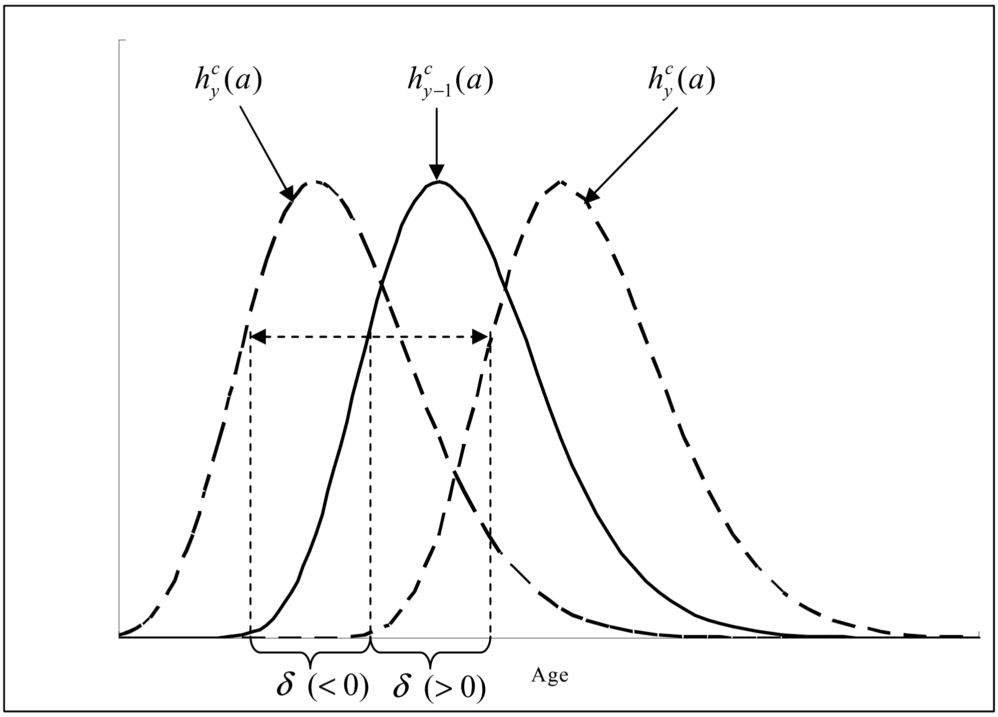
In continuous form, equation (5.8) can be written as

$$G_t = \int_{15}^{50} h_{t-50}^c((1+\delta)\cdot a - 49\cdot\delta)da \tag{5.9}$$

Let $u = (1+\delta)\cdot a - 49\cdot\delta$, then we have $da = du/(1+\delta)$, where $\delta \neq -1$. Therefore, equation (5.9) becomes

$$G_t = \int_{15}^{50} h_{t-50}^c((1+\delta)\cdot a - 49\cdot\delta)da = \frac{1}{1+\delta}\cdot\int_{15}^{50} h_{t-50}^c(u)du = \frac{1}{1+\delta} = 1 - \frac{\delta}{1+\delta} \quad (5.10)$$

Figure 8. Parallel shifting of the standardized cohort fertility curves along the age-axis



Equation (5.10) shows that G_t is a decreasing function of δ (i.e. the larger the δ, the smaller the G_t). Equation (5.9) also shows that (i) when all the concerned standardized cohort fertility curves are exactly the same (i.e. $\delta = 0$), we have $G_t = 1$; (ii) when the concerned standardized cohort fertility curves shift to the right (i.e. to higher ages, but with no change in the shape) by the same amount (i.e. $\delta > 0$) from one cohort to the next (i.e. women postpone childbearing), we have $G_t < 1$; (iii). when the concerned standardized cohort fertility curves shift to the left (i.e. to lower ages, but with no change in the shape) by the same amount (i.e. $\delta < 0$) from one cohort to the next (i.e. women advance childbearing), we have $G_t > 1$. For example, if $\delta = 0.1$, then $G_t = 0.91$; if $\delta = -0.1$, then $G_t = 1.11$.

Equation (5.10) also shows that G_t is a non-linear function of δ (i.e. a hyperbola with $\delta \neq -1$). But when $-0.10 \le \delta \le 0.10$, G_t is very close to a linear function of δ (with the coefficient of determination $R^2 = 0.997$), i.e. on the interval [-0.10, 0.10], we have $G_t \approx 1 - \delta$ (Actually, the power series expansion can be considered: $\frac{1}{1+\delta} = 1-\delta+\delta^2-\delta^3+\delta^4-\cdots$, where $|\delta|<1$). Since $[\mu\langle h_y^c\rangle]'_y = \delta$, we have $G_t \approx 1-[\mu\langle h_y^c\rangle]'_y$. Here, we notice that G_t is determined by the rate of change ($[\mu\langle h_y^c\rangle]'_y$) in the mean age of the standardized cohort fertility schedule ($h_y^c(a)$) almost in the same form, under the two different assumptions (i.e. I and II).

Under a stricter assumption (i.e. constant cohort and period quanta), Zeng and Land (2002) obtained a similar result as the one expressed in equation (5.9), who used the symbol r_c in their paper. However, our analysis above shows that the assumption of constant cohort and period quanta is not necessary for the result to hold.

Now, let's take a look at the relationship between the mean ages of the standardized period and cohort fertility curves. From the discussions above, we have

$$\mu\langle H_t^p\rangle = \sum_{a=15}^{49}[a\cdot H_t^p(a)] = \frac{1}{G_t}\cdot\sum_{a=15}^{49}[a\cdot h_t^p(a)] = \frac{1}{G_t}\cdot\sum_{a=15}^{49}[a\cdot h_{t-(a+1)}^c(a)]$$

$$= \frac{1}{G_t}\cdot\sum_{a=15}^{49}[a\cdot h_{t-50}^c((1+\delta)\cdot a-49\cdot\delta)] \tag{5.11}$$

In continuous form, equation (5.11) can be written as

$$\mu\langle H_t^p\rangle = \frac{1}{G_t}\cdot\int_{15}^{50}[a\cdot h_{t-50}^c((1+\delta)\cdot a-49\cdot\delta)]da \tag{5.12}$$

Let $u=(1+\delta)\cdot a-49\cdot\delta$, then we have $a=(u+49\cdot\delta)/(1+\delta)$ and $da = du/(1+\delta)$, where $\delta\neq -1$. Therefore, equation (5.12) can be written as

$$\mu\langle H_t^p\rangle = \frac{1}{G_t\cdot(1+\delta)^2}\cdot\int_{15}^{50}[(u+49\cdot\delta)\cdot h_{t-50}^c(u)]du \tag{5.13}$$

Since $G_t = 1/(1+\delta)$, equation (5.13) becomes

$$\mu\langle H_t^p\rangle = \frac{1}{1+\delta}\cdot\int_{15}^{50}[(u+49\cdot\delta)\cdot h_{t-50}^c(u)]du = \frac{1}{1+\delta}\cdot\left[\int_{15}^{50}[u\cdot h_{t-50}^c(u)]du + 49\cdot\delta\cdot\int_{15}^{50}h_{t-50}^c(u)du\right]$$

$$= \frac{1}{1+\delta}\cdot(\mu\langle h_{t-50}^c\rangle + 49\cdot\delta) = \mu\langle h_{t-50}^c\rangle + \frac{\delta}{1+\delta}\cdot(49-\mu\langle h_{t-50}^c\rangle) \tag{5.14}$$

where $\mu\langle h_{t-50}^c\rangle = \int_{15}^{50}[u\cdot h_{t-50}^c(u)]du$ is the mean age of the standardized fertility schedule of the benchmark birth cohort (i.e. birth cohort t-50). Taking the first derivative with respect to δ on both sides of equation (5.14), we obtain $[\mu\langle H_t^p\rangle]'_\delta = (49-\mu\langle h_{t-50}^c\rangle)\cdot\frac{1}{(1+\delta)^2} > 0$. Therefore, $\mu\langle H_t^p\rangle$ is an increasing function of δ (i.e. the larger the δ, the lager the $\mu\langle H_t^p\rangle$).

Since $\mu\langle h_{t-16}^c\rangle = \mu\langle h_{t-50}^c\rangle + 34\cdot\delta$, it follows from equation (5.14)

$$\mu\langle H_t^p\rangle = \frac{1}{1+\delta}\cdot(\mu\langle h_{t-16}^c\rangle + 15\cdot\delta) = \mu\langle h_{t-16}^c\rangle - \frac{\delta}{1+\delta}\cdot(\mu\langle h_{t-16}^c\rangle - 15) \tag{5.15}$$

where $\mu\langle h_{t-16}^c\rangle = \int_{15}^{50}[u\cdot h_{t-16}^c(u)]du$ is the mean age of the standardized age-specific fertility curve of the birth cohort t-16.

From equations (5.14) and (5.15), it is obvious that (i) if $\delta = 0$, then $\mu\langle H_t^p\rangle = \mu\langle h_{t-50}^c\rangle$, (ii) if $\delta > 0$, then $\mu\langle h_{t-50}^c\rangle < \mu\langle H_t^p\rangle < \mu\langle h_{t-16}^c\rangle$, (iii) if $\delta < 0$, then $\mu\langle h_{t-16}^c\rangle < \mu\langle H_t^p\rangle < \mu\langle h_{t-50}^c\rangle$. For example, assume $\mu\langle h_{t-50}^c\rangle = 28$, then $\mu\langle h_{t-16}^c\rangle = 28 + 34\cdot\delta$. Therefore, if $\delta = 0.1$, then $\mu\langle h_{t-16}^c\rangle = 31.40$ and $\mu\langle H_t^p\rangle = 29.91$; if $\delta = -0.1$, then $\mu\langle h_{t-16}^c\rangle = 24.60$ and $\mu\langle H_t^p\rangle = 25.67$.

Next, let's take a look at the relationship between the variances of the standardized period and cohort fertility curves. From the discussions above, we have

$$v\langle H_t^p\rangle = \sum_{a=15}^{49}[(a - H\langle h_t^p\rangle)^2\cdot H_t^p(a)] = \frac{1}{G_t}\cdot\sum_{a=15}^{49}[(a-\mu\langle H_t^p\rangle)^2\cdot h_t^p(a)]$$

$$= \frac{1}{G_t}\cdot\sum_{a=15}^{49}[(a-\mu\langle H_t^p\rangle)^2\cdot h_{t-(a+1)}^c(a)]$$

$$= \frac{1}{G_t}\cdot\sum_{a=15}^{49}[(a-\mu\langle H_t^p\rangle)^2\cdot h_{t-50}^c((1+\delta)\cdot a - 49\cdot\delta)] \tag{5.16}$$

In continuous form, equation (5.16) can be written as

$$v\langle H_t^p\rangle = \frac{1}{G_t}\cdot\int_{15}^{50}[(a-\mu\langle H_t^p\rangle)^2\cdot h_{t-50}^c((1+\delta)\cdot a - 49\cdot\delta)]da \tag{5.16}$$

Let $u = (1+\delta)\cdot a - 49\cdot\delta$, then we have $a = (u + 49\cdot\delta)/(1+\delta)$ and $da = du/(1+\delta)$, where $\delta \neq -1$. Therefore, equation (5.16) can be written as

$$v\left\langle H_t^p \right\rangle = \frac{1}{G_t \cdot (1+\delta)} \cdot \int_{15}^{50} \left[\left(\frac{u + 49\cdot\delta}{1+\delta} - \mu\left\langle H_t^p \right\rangle \right)^2 \cdot h_{t-50}^c(u) \right] du$$

$$= \frac{1}{G_t \cdot (1+\delta)} \cdot \int_{15}^{50} \left[\left(\frac{u + 49\cdot\delta}{1+\delta} - \frac{\mu\left\langle h_{t-50}^c \right\rangle + 49\cdot\delta}{1+\delta} \right)^2 \cdot h_{t-50}^c(u) \right] du$$

$$= \frac{1}{G_t \cdot (1+\delta)} \cdot \int_{15}^{50} \left[\left(\frac{u - \mu\left\langle h_{t-50}^c \right\rangle}{1+\delta} \right)^2 \cdot h_{t-50}^c(u) \right] du \tag{5.18}$$

Since $G_t = 1/(1+\delta)$, equation (5.18) becomes

$$v\left\langle H_t^p \right\rangle = \frac{1}{(1+\delta)^2} \cdot \int_{15}^{50} [(u - \mu\left\langle h_{t-50}^c \right\rangle)^2 \cdot h_{t-50}^c(u)] du = \frac{1}{(1+\delta)^2} \cdot v\left\langle h_{t-50}^c \right\rangle = \left(\frac{\sigma\left\langle h_{t-50}^c \right\rangle}{1+\delta} \right)^2 \tag{5.18}$$

Equivalently, we have $\sigma\left\langle H_t^p \right\rangle = \sigma\left\langle h_{t-50}^c \right\rangle / (1+\delta)$, where σ stands for standard deviation. Equation (5.18) shows that $v\left\langle H_t^p \right\rangle$ is a decreasing function of δ (i.e. the larger the δ, the smaller the $v\left\langle H_t^p \right\rangle$). It is obvious that (i) if $\delta = 0$, then $\sigma\left\langle H_t^p \right\rangle = \sigma\left\langle h_{t-50}^c \right\rangle$; (ii) if $\delta > 0$, then $\sigma\left\langle H_t^p \right\rangle < \sigma\left\langle h_{t-50}^c \right\rangle$; (iii) if $\delta < 0$, then $\sigma\left\langle H_t^p \right\rangle > \sigma\left\langle h_{t-50}^c \right\rangle$. For example, assume $\delta\left\langle h_{t-50}^c \right\rangle = 4$, then if $\delta = 0.1$, then $\sigma\left\langle H_t^p \right\rangle = 3.64$; if $\delta = -0.1$, then $\sigma\left\langle H_t^p \right\rangle = 4.44$.

Similarly, it can be proved that

$$s\left\langle H_t^p \right\rangle = \int_{15}^{50} \left[\left(\frac{a - \mu\left\langle H_t^p \right\rangle}{\sigma\left\langle H_t^p \right\rangle} \right)^3 \cdot H_t^p(a) \right] da = s\left\langle h_{t-50}^c \right\rangle \tag{5.20}$$

$$k\left\langle H_t^p \right\rangle = \int_{15}^{50} \left[\left(\frac{a - \mu\left\langle H_t^p \right\rangle}{\sigma\left\langle H_t^p \right\rangle} \right)^4 \cdot H_t^p(a) \right] da = k\left\langle h_{t-50}^c \right\rangle \tag{5.21}$$

where *s* and *k* stand for skewness and kurtosis respectively.

Now, we discuss the relationship between the period and the cohort mean ages. It is obvious that birth cohort $y = t - (\mu\left\langle H_t^p \right\rangle + 1)$ is aged $\mu\left\langle H_t^p \right\rangle$ (when $\mu\left\langle H_t^p \right\rangle$ is an integer) at the beginning of year *t*. From equations (5.7) and (5.14), we have

$$\mu\left\langle h^c_{t-(\mu\langle H_t^p\rangle+1)}\right\rangle = \mu\left\langle h^c_{t-50}\right\rangle + [t-(\mu\left\langle H_t^p\right\rangle+1)-(t-50)]\cdot\delta = \mu\left\langle h^c_{t-50}\right\rangle + [49-\mu\left\langle H_t^p\right\rangle]\cdot\delta$$

$$= \mu\left\langle h^c_{t-50}\right\rangle + \left(49 - \frac{\mu\left\langle h^c_{t-50}\right\rangle + 49\cdot\delta}{1+\delta}\right)\cdot\delta = \mu\left\langle h^c_{t-50}\right\rangle + \frac{\delta}{1+\delta}\cdot(49-\mu\left\langle h^c_{t-50}\right\rangle)$$

$$= \mu\left\langle H_t^p\right\rangle \tag{5.22}$$

The above discussion indicates that, under the assumption stated, the mean age of the standardized period fertility schedule $H_t^p(a)$ is the same as the mean age of the standardized fertility schedule of the birth cohort $y = t-(\mu\left\langle H_t^p\right\rangle+1)$ that reaches its mean age of fertility right at the beginning of year *t*.

6. Relationship between the two period curves $f_t^p(a)$ and $h_t^p(a)$

In the discussions above, there are two important period fertility curves for year *t*, i.e. $f_t^p(a)$ and $h_t^p(a)$, whose general relationship is given in equation (2.4). Now, we will look at the relationships between the positions and shapes of the two period curves.

Let $F_t^p(a) = f_t^p(a)/TFR_t$, $H_t^p(a) = h_t^p(a)/G_t$, then we have (i) $F_t^p(a) \geq 0$, $\sum_{a=15}^{49} F_t^p(a) = 1$, and (ii) $H_t^p(a) \geq 0$, $\sum_{a=15}^{49} H_t^p(a) = 1$. Hence, equation (2.4) can be rewritten as

$$F_t^p(a) = \frac{G_t}{TFR_t}\cdot H_t^p(a)\cdot LFR_{t-(a+1)} = \frac{G_t}{TFR_t}\cdot H_t^p(a)\cdot LFR_{T-a} \tag{6.1}$$

where $T = t-1$. Suppose that LFR_y can be expressed by the following n^{th}-degree polynomial of *y*:

$$LFR_y = \lambda_0 + \lambda_1\cdot y + \lambda_2\cdot y^2 + \cdots + \lambda_n\cdot y^n = \sum_{i=0}^{n}\left(\lambda_i\cdot y^i\right) \tag{6.2}$$

where *n* is a non-negative integer and λ_i, $(i = 0, 1, 2, \ldots, n)$, are the polynomial coefficients. Then equation (6.1) becomes

$$F_t^p(a) = \frac{G_t}{TFR_t}\cdot H_t^p(a)\cdot\sum_{i=0}^{n}[\lambda_i\cdot(T-a)^i] \tag{6.3}$$

Applying the binomial theorem on $(T-a)^i$, we have

$$F_t^p(a) = \frac{G_t}{TFR_t} \cdot H_t^p(a) \cdot \sum_{i=0}^{n} \left\{ \lambda_i \cdot \sum_{j=0}^{i} \left[(-1)^j \cdot \left(\frac{i!}{j! \cdot (i-j)!} \right) \cdot T^{i-j} \cdot a^j \right] \right\}$$

$$= \frac{G_t}{TFR_t} \cdot \sum_{i=0}^{n} \left\{ \lambda_i \cdot \sum_{j=0}^{i} \left[(-1)^j \cdot \left(\frac{i!}{j! \cdot (i-j)!} \right) \cdot T^{i-j} \cdot [a^j \cdot H_t^p(a)] \right] \right\} \quad (6.4)$$

Define the r^{th} absolute moment (about zero or origin) of $F_t^p(a)$ and $H_t^p(a)$ as follows:

$$\hat{M}_r \langle F_t^p \rangle = \sum_{a=15}^{49} [a^r \cdot F_t^p(a)], \quad r = 0, 1, 2, \ldots \quad (6.5)$$

$$\hat{M}_r \langle H_t^p \rangle = \sum_{a=15}^{49} [a^r \cdot H_t^p(a)], \quad r = 0, 1, 2, \ldots \quad (6.6)$$

Then, from equation (6.4), we have

$$\hat{M}_r \langle F_t^p \rangle = \frac{G_t}{TFR_t} \cdot \sum_{i=0}^{n} \left\{ \lambda_i \cdot \sum_{j=0}^{i} \left[(-1)^j \cdot \left(\frac{i!}{j! \cdot (i-j)!} \right) \cdot T^{i-j} \cdot \sum_{a=15}^{49} [a^{j+r} \cdot H_t^p(a)] \right] \right\}$$

$$= \frac{G_t}{TFR_t} \cdot \sum_{i=0}^{n} \left\{ \lambda_i \cdot \sum_{j=0}^{i} \left[(-1)^j \cdot \left(\frac{i!}{j! \cdot (i-j)!} \right) \cdot T^{i-j} \cdot \hat{M}_{j+r} \langle H_t^p \rangle \right] \right\} \quad (6.7)$$

Equation (6.7) provides a general expression for the relationship between the absolute moments of $F_t^p(a)$ and $H_t^p(a)$.

Now, let's consider one specific case, where LFR_y is a linear function of *y*. In this case, we have. $LFR_y = \lambda_0 + \lambda_1 \cdot y$. Therefore, equation (6.7) simplifies to

$$\hat{M}_r \langle F_t^p \rangle = \frac{G_t}{TFR_t} \cdot \left\{ (\lambda_0 + \lambda_1 \cdot T) \cdot \sum_{a=15}^{49} [a^r \cdot H_t^p(a)] - \lambda_1 \cdot \sum_{a=15}^{49} [a^{r+1} \cdot H_t^p(a)] \right\}$$

$$= \frac{G_t}{TFR_t} \cdot [(\lambda_0 + \lambda_1 \cdot T) \cdot \hat{M}_r \langle H_t^p \rangle - \lambda_1 \cdot \hat{M}_{r+1} \langle H_t^p \rangle] \quad (6.8)$$

Specially, the mean age of $F_t^p(a)$ is as follows:

$$\mu \langle F_t^p \rangle = \sum_{a=15}^{49} [a \cdot F_t^p(a)] = \frac{G_t}{TFR_t} \cdot [(\lambda_0 + \lambda_1 \cdot T) \cdot \hat{M}_1 \langle H_t^p \rangle - \lambda_1 \cdot \hat{M}_2 \langle H_t^p \rangle]$$

$$= \frac{G_t}{TFR_t} \cdot [(\lambda_0 + \lambda_1 \cdot T) \cdot \mu \langle H_t^p \rangle - \lambda_1 \cdot (v \langle H_t^p \rangle + (\mu \langle H_t^p \rangle)^2)]$$

$$= \frac{G_t}{TFR_t} \cdot [(\lambda_0 \cdot \mu \langle H_t^p \rangle + \lambda_1 \cdot (T \cdot \mu \langle H_t^p \rangle - (\mu \langle H_t^p \rangle)^2 - v \langle H_t^p \rangle))] \quad (6.9)$$

Since $LFR_y = \lambda_0 + \lambda_1 \cdot y$, we have $LFR_y - LFR_{y-1} = \lambda_1$ $(y = t-49, t-48, \ldots, t-16)$, and therefore,

$$LFR_y = LFR_{t-50} + \lambda_1 \cdot [y - (t-50)] \qquad (6.10)$$

Based on equation (3.3), equation (6.1) can be rewritten as

$$F_t^p(a) = \frac{G_t}{TFR_t} \cdot H_t^p(a) \cdot LFR_{t-(a+1)} = \left(\frac{LFR_{t-(a+1)}}{LFR_{t-(\mu\langle H_t^p \rangle + 1)}} \right) \cdot H_t^p(a) \qquad (6.11)$$

It is obvious from equation (6.11) that $F_t^p(\mu\langle H_t^p \rangle) = H_t^p(\mu\langle H_t^p \rangle)$. Further

$$F_t^p(a) - H_t^p(a) == \left(\frac{LFR_{t-(a+1)} - LFR_{t-(\mu\langle H_t^p \rangle + 1)}}{LFR_{t-(\mu\langle H_t^p \rangle + 1)}} \right) \cdot H_t^p(a) \qquad (6.12)$$

From equations (6.10) and (6.12), we obtain

$$F_t^p(a) - H_t^p(a) == \lambda_1 \cdot \left(\frac{\mu\langle H_t^p \rangle - a}{LFR_{t-(\mu\langle H_t^p \rangle + 1)}} \right) \cdot H_t^p(a) = \frac{\lambda_1}{LFR_{t-(\mu\langle H_t^p \rangle + 1)}} \cdot (\mu\langle H_t^p \rangle - a) \cdot H_t^p(a) \qquad (6.13)$$

From equation (6.13), we obtain

$$\sum_{a=15}^{49} [a \cdot F_t^p(a)] - \sum_{a=15}^{49} [a \cdot H_t^p(a)] = \frac{\lambda_1}{LFR_{t-(\mu\langle H_t^p \rangle + 1)}} \cdot \sum_{a=15}^{49} [(\mu\langle H_t^p \rangle \cdot a - a^2) \cdot H_t^p(a)] \qquad (6.14)$$

that is

$$\mu\langle F_t^p \rangle - \mu\langle H_t^p \rangle = \frac{\lambda_1}{LFR_{t-(\mu\langle H_t^p \rangle + 1)}} \cdot \sum_{a=15}^{49} [(\mu\langle H_t^p \rangle \cdot a - a^2) \cdot H_t^p(a)]$$

$$= \frac{\lambda_1}{LFR_{t-(\mu\langle H_t^p \rangle + 1)}} \cdot \left[\mu\langle H_t^p \rangle \cdot \sum_{a=15}^{49} [(\cdot a \cdot H_t^p(a)] - \sum_{a=15}^{49} a^2 \cdot H_t^p(a) \right]$$

$$= \frac{\lambda_1}{LFR_{t-(\mu\langle H_t^p \rangle + 1)}} \cdot \left[(\mu\langle H_t^p \rangle)^2 - \sum_{a=15}^{49} a^2 \cdot H_t^p(a) \right] \qquad (6.14)$$

$$= -\lambda_1 \cdot \left(\frac{v\langle H_t^p \rangle}{LFR_{t-(\mu\langle H_t^p \rangle + 1)}} \right) \qquad (6.14)$$

Therefore, when $\lambda_1 > 0$, we have $\mu\langle F_t^p \rangle < \mu\langle H_t^p \rangle$, when $\lambda_1 < 0$, we have $\mu\langle F_t^p \rangle > \mu\langle H_t^p \rangle$.

Based on equation (6.10), we can rewrite equation (6.14) as

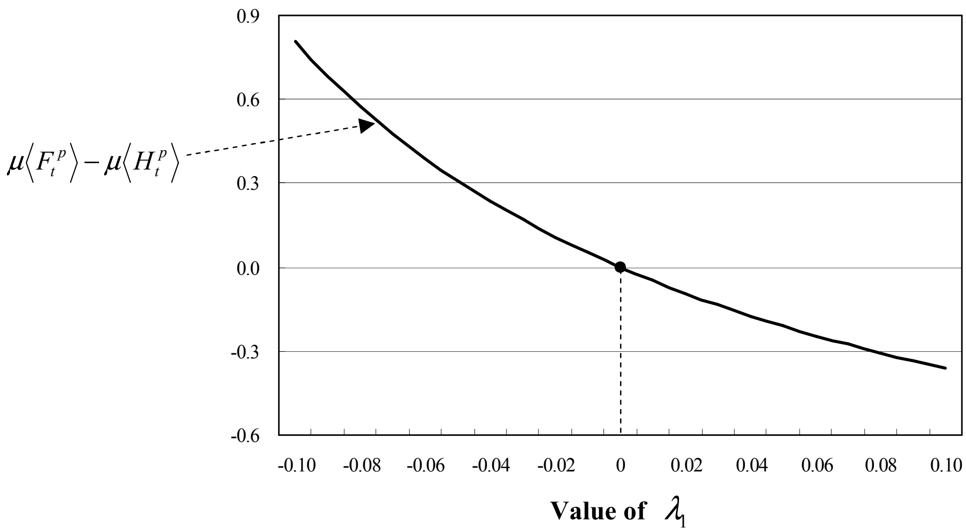
$$\mu\langle F_t^p\rangle - \mu\langle H_t^p\rangle = -\frac{\lambda_1 \cdot v\langle H_t^p\rangle}{LFR_{t-50} + \lambda_1 \cdot (49 - \mu\langle H_t^p\rangle)} \tag{6.15}$$

Equation (6.15) also shows that $\mu\langle F_t^p\rangle - \mu\langle H_t^p\rangle$ is a hyperbolic function of λ_1. Taking the first derivative with respect to λ_1 on both sides of equation (6.15), we obtain

$$[\mu\langle F_t^p\rangle - \mu\langle H_t^p\rangle]'_{\lambda_1} = -\frac{v\langle H_t^p\rangle \cdot LFR_{t-50}}{[LFR_{t-50} + \lambda_1 \cdot (49 - \mu\langle H_t^p\rangle)]^2} < 0 \tag{6.16}$$

Therefore, $\mu\langle F_t^p\rangle - \mu\langle H_t^p\rangle$ is a decreasing function of λ_1. In other words, $\left|\mu\langle F_t^p\rangle - \mu\langle H_t^p\rangle\right|$ is an increasing function of $|\lambda_1|$. Figure 9 graphs the relationship between $\mu\langle F_t^p\rangle - \mu\langle H_t^p\rangle$ and λ_1 (assuming $LFR_{t-50} = 5$, $\mu\langle H_t^p\rangle = 30$, and $v\langle H_t^p\rangle = 25$).

Figure 9. Relationship between $\mu\langle F_t^p\rangle - \mu\langle H_t^p\rangle$ and λ_1



7. A closer examination of the Ryder's basic translation equation

In his classic paper on demographic translation, Ryder (1964) developed the following basic translation equation between period total fertility rate and cohort total fertility rate:

$$B(0, T + \mu_1) = [\beta(0, T)] \cdot [1 - \mu_1'(T)] \tag{7.1}$$

In normal term, equation (7.1) is equivalent to

$$TFR_{T+\mu_1} = LFR_T \cdot [1 - \mu_1'(T)] \tag{7.2}$$

Ryder arrived at the relationship in equation (7.1) based on the assumption that for each age, the time series of the age-specific fertility rates may be represented by an n^{th}-degree polynomial with respect to *T*, where *T* denotes the birth cohort (i.e. year of birth).

Following the approach of Ryder (1964), we assume that for each age $a \quad (a = 15, 16, \ldots, 49)$, the time series $\{ f_y^c(a) \mid y = t-50, t-49, \ldots, t-16 \}$ can be represented by the following n^{th}-degree polynomial of *y*:

$$f_y^c(a) = \rho_0(a) + \rho_1(a) \cdot y + \rho_2(a) \cdot y^2 + \cdots + \rho_n(a) \cdot y^n = \sum_{i=0}^{n} [\rho_i(a) \cdot y^i] \tag{7.3}$$

where *n* is a non-negative integer and $\rho_i(a)$, $i = 0, 1, 2, \ldots, n$, are the polynomial coefficients.

Then, from equation (2.2), we obtain the cohort total fertility rate:

$$LFR_y = \sum_{a=15}^{49} f_y^c(a) = \sum_{a=15}^{49} \left\{ \sum_{i=0}^{n} [\rho_i(a) \cdot y^i] \right\} = \sum_{i=0}^{n} \left[\left(\sum_{a=15}^{49} \rho_i(a) \right) \cdot y^i \right] \tag{7.4}$$

It is obvious from equation (7.4) that under the assumption expressed in equation (7.3), the cohort total fertility rate (i.e. LFR_y) is also an n^{th}-degree polynomial with respect to *y*.

Similarly, from equation (2.5), we obtain the period total fertility rate:

$$TFR_t = \sum_{a=15}^{49} f_{t-(a+1)}^c(a) = \sum_{a=15}^{49} \left(\sum_{i=0}^{n} [\rho_i(a) \cdot (t-(a+1))^i] \right) = \sum_{a=15}^{49} \left(\sum_{i=0}^{n} [\rho_i(a) \cdot (T-a)^i] \right) \tag{7.5}$$

where $T = t - 1$. Using the binomial theorem, we have

$$TFR_t = \sum_{a=15}^{49} \left\{ \sum_{i=0}^{n} \left[\sum_{j=0}^{i} \left((-1)^j \cdot \left(\frac{i!}{j! \cdot (i-j)!} \right) \cdot T^{i-j} \cdot a^j \cdot \rho_i(a) \right) \right] \right\} \tag{7.6}$$

It is obvious from equation (7.6) that under the assumption expressed in equation (7.3), the period total fertility rate (i.e. TFR_t) is an n^{th}-degree polynomial with respect to *t*.

Under the assumption expressed in equation (7.3), we have the mean age of childbearing of birth cohort *y*:

$$\mu\langle f_y^c \rangle = \sum_{a=15}^{49} \left[a \cdot \left(\frac{f_y^c(a)}{LFR_y} \right) \right] = \sum_{a=15}^{49} [a \cdot h_y^c(a)] \tag{7.7}$$

where $h_y^c(a) = f_y^c(a) \big/ LFR_y$. In this case, it is obvious that $h_y^c(a)$ is a ratio of two n^{th}-degree polynomials with respect to *y*. Therefore, even the first derivative of $\mu\langle f_y^c \rangle$ is a very complex function of *y*.

Now, we look at a very special situation. Let's assume that LFR_y is a constant (denoted as LFR), then equation (7.4) becomes

$$\sum_{i=0}^{n}\left[\left(\sum_{a=15}^{49}\rho_i(a)\right)\cdot y^i\right] = LFR \tag{7.8}$$

Taking the n^{th} derivative on both sides of equation (7.8), we have $n!\cdot\sum_{a=15}^{49}\rho_n(a) = 0$. Therefore, $\sum_{a=15}^{49}\rho_n(a) = 0$. Similarly, it can be proved that $\sum_{a=15}^{49}\rho_{n-1}(a) = 0$, …, $\sum_{a=15}^{49}\rho_1(a) = 0$. And finally, $\sum_{a=15}^{49}\rho_0(a) = LFR$. In this case, equation (7.7) becomes

$$\mu\langle f_y^c \rangle = \frac{1}{LFR}\cdot\sum_{a=15}^{49}[a\cdot f_y^c(a)] = \frac{1}{LFR}\cdot\sum_{a=15}^{49}\left\{a\cdot\sum_{i=0}^{n}[\rho_i(a)\cdot y^i]\right\}$$
$$= \frac{1}{LFR}\cdot\sum_{i=0}^{n}\left\{\left(\sum_{a=15}^{49}[a\cdot\rho_i(a)]\right)\cdot y^i\right\} \tag{7.9}$$

Equation (7.9) shows that under the assumptions stated, $\mu\langle f_y^c \rangle$ is an n^{th}-degree polynomial with respect to *y*. It follows that the first derivative of $\mu\langle f_y^c \rangle$ with respect to *y* is an $(n\text{-}1)^{\text{th}}$ polynomial of *y*. Therefore, the first derivative of $\mu\langle f_y^c \rangle$ is constant if and only if $n = 1$.

Under the assumptions that (i) for each age *a* $(a = 15, 16, \ldots, 49)$, the time series $f_y^c(a)$, $y = t\text{-}50$, $t\text{-}49$, …, $t\text{-}16$, can be represented by a linear function of *y*, and (ii) LFR_y is constant with respect to *y* (denoted as LFR), we have from equations (7.4) and (7.5)

$$LFR = \sum_{a=15}^{49}\rho_0(a) \tag{7.10}$$

$$TFR_t = \sum_{a=15}^{49}\rho_0(a) + \sum_{a=15}^{49}[\rho_1(a)\cdot(T-a)] = \sum_{a=15}^{49}\rho_0(a) - \sum_{a=15}^{49}[a\cdot\rho_1(a)] \tag{7.11}$$

Combining equations (7.10) and (7.11), we obtain

$$TFR_t = LFR - \sum_{a=15}^{49}[a \cdot \rho_1(a)] \tag{7.12}$$

From equation (7.9), we have

$$\mu\langle f_y^c \rangle = \frac{1}{LFR} \cdot \left[\sum_{a=15}^{49}[a \cdot \rho_0(a)] + \left(\sum_{a=15}^{49}[a \cdot \rho_1(a)] \right) \cdot y \right] \tag{7.13}$$

Equation (7.13) shows that, under the assumptions stated above, $\mu\langle f_y^c \rangle$ is a linear function of *y*. Taking the first derivative with respect to *y* on both sides of equation (7.13), we get

$$[\mu\langle f_y^c \rangle]_y^{'} = \frac{1}{LFR} \cdot \sum_{a=15}^{49}[a \cdot \rho_1(a)] \tag{7.14}$$

Equation (7.14) shows that, under the assumptions stated above, $[\mu\langle f_y^c \rangle]_y^{'}$ is constant with respect to *y* and implies that $\sum_{a=15}^{49}[a \cdot \rho_1(a)] = LFR \cdot [\mu\langle f_y^c \rangle]_y^{'}$. Consequently, equation (7.12) becomes

$$TFR_t = LFR \cdot (1 - [\mu\langle f_y^c \rangle]_y^{'}) \tag{7.15}$$

Equation (7.15) shows that, under the assumptions stated above, TFR_t is constant with respect to *t*.

8. Effect of change in the cohort standard deviation on G_t

Mathematically, it is very complex to investigate, in a general way, the effect of change in the cohort standard deviation on G_t. Therefore, we have to assume that the standardized cohort fertility schedule (i.e. $h_y^c(a)$) follow certain continuous probability distribution. For this purpose, we will use the following Gamma function for the simulation.

$$g(a) = \begin{cases} K \cdot (a - a_0)^A \cdot e^{-B \cdot (a - a_0)}, & when \quad a \geq a_0 \\ 0, & when \quad a < a_0 \end{cases} \tag{8.1}$$

The properties of the above-defined Gamma function are discussed in detail in Annex B.

In order to examine the effect of change (increment/decrement, denoted as $\Delta\sigma\langle g \rangle$) in the cohort standard deviation on G_t, we calculated the corresponding values of G_t using the above Gamma distribution. In this connection, three scenarios were simulated as follows:

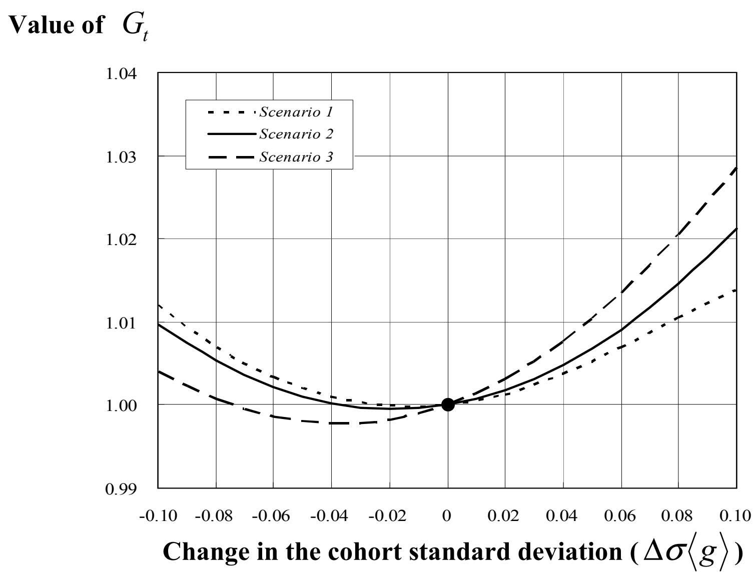
Scenario 1: The mean age of fertility ($\mu\langle g\rangle$) is held constant at 26 for all birth cohorts and the standard deviation of the benchmark cohort (i.e. the start cohort) is set to be 5.

Scenario 2: The mean age of fertility ($\mu\langle g\rangle$) is held constant at 30 for all birth cohorts and the standard deviation of the benchmark cohort (i.e. the start cohort) is set to be 5.

Scenario 3: The mean age of fertility ($\mu\langle g\rangle$) is held constant at 34 for all birth cohorts and the standard deviation of the benchmark cohort (i.e. the start cohort) is set to be 5.

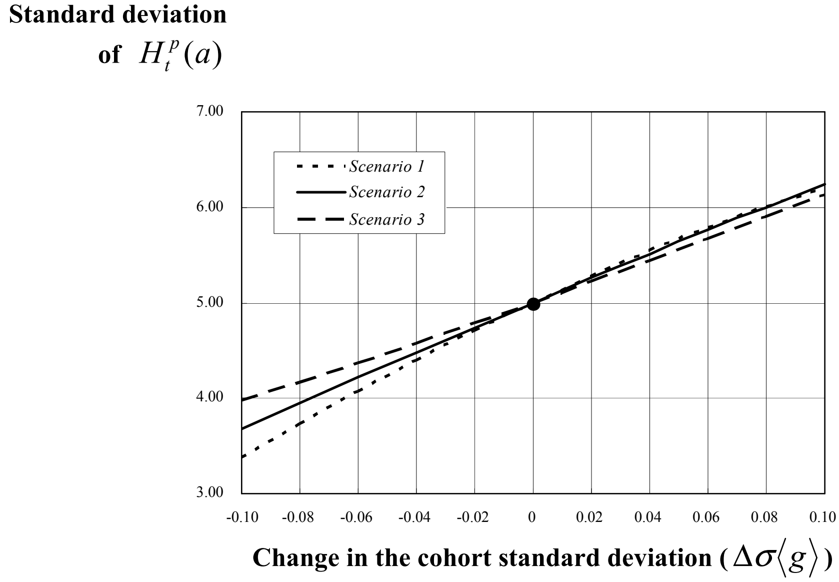
The results of the numerical simulations are shown in Figure 10. From the results of the three scenarios, we notice that (i) if the change in the cohort standard deviation is positive, then $G_t > 1$; (ii) the higher the mean age of fertility, the larger the effect of change in the cohort standard deviation on G_t is. The numerical simulations also show that when $26 \le \mu\langle g\rangle \le 34$ and $-0.10 \le \Delta\sigma\langle g\rangle \le 0.10$, we have $0.99 < G_t < 1.03$. Therefore, based on the numerical simulations, it is plausible to conclude that the effect of change in the cohort standard deviation on G_t is basically negligible. In terms of the shape of the curves depicted in Figure 10, they are close to parabolas on the interval [-0.10, 0.10] of $\Delta\sigma\langle g\rangle$. If the following quadratic function $G_t = 1 + \varphi_1 \cdot \Delta\sigma\langle g\rangle + \varphi_2 \cdot (\Delta\sigma\langle g\rangle)^2$, where φ_1 and φ_2 are coefficients, is used to fit the curves, then we have $R^2 > 0.992$ (i.e. the coefficient of determination) for all the three scenarios.

Figure 10. Effect of change in the cohort standard deviation on G_t



Similarly, we simulated the effects of change in the cohort standard deviation on the standard deviation of $H_t^p(a)$. Figure 11 shows the results for the three scenarios. It is clear that, when $26 \le \mu\langle g\rangle \le 34$ and $-0.10 \le \Delta\sigma\langle g\rangle \le 0.10$, the standard deviation of $H_t^p(a)$ is a monotonically increasing function of $\Delta\sigma\langle g\rangle$ when the cohort mean age of fertility ($\mu\langle g\rangle$) is held constant. In terms of the shape of the curves depicted in Figure 11, they are close to straight lines on the interval [-0.10, 0.10] of $\Delta\sigma\langle g\rangle$, with the coefficient of determination $R^2 > 0.993$ for all the three scenarios.

Figure 11. Effect of change in the cohort standard deviation on the standard deviation of $H_t^p(a)$



9. Suggestion for further study

In this paper, the relationships between period and cohort fertility are examined in various ways from a cohort-to-period perspective. Similarly, the relationships could also be examined from a period-to-cohort perspective. To gain further insights into the relationships, more numerical simulations and empirical analyses could be conducted.

Annex A. Some properties of the (statistical) moments

In this annex, we discuss some properties of the (statistical) moments that are relevant to the present paper. Suppose that function $p(x)$ represents a probability distribution (i.e. $p(x)$ satisfies $p(x) \geq 0$ and $\sum_{all\ x} p(x) = 1$), then its moments are defined as follows:

The r^{th} absolute moment (about zero or origin) of $p(x)$ is defined as

$$\hat{M}_r\langle p\rangle = \sum_{all\ x}[x^r \cdot p(x)] \tag{A.1}$$

where r is a non-negative integer. It is obvious that $\hat{M}_0\langle p\rangle = 1$ and $\hat{M}_1\langle p\rangle = \mu\langle p\rangle$, where $\mu\langle p\rangle = \sum_{all\ x}[x \cdot p(x)]$ is the mean of $p(x)$.

The r^{th} central moment (about mean) of $p(x)$ is defined as

$$\tilde{M}_r\langle p\rangle = \sum_{all\ x}[(x - \mu\langle p\rangle)^r \cdot p(x)] \tag{A.2}$$

where r is a non-negative integer. It is obvious that $\tilde{M}_0\langle p\rangle = 1$, $\tilde{M}_1\langle p\rangle = 0$, and $\tilde{M}_2\langle p\rangle = v\langle p\rangle$, where $v\langle p\rangle = \sum_{all\ x}[(x - \mu\langle p\rangle)^2 \cdot p(x)]$ is the variance of $p(x)$.

The absolute and the central moments are important statistical measures for describing the position and the shape of a probability distribution.

By the binomial theorem, we have

$$\tilde{M}_r\langle p\rangle = \sum_{all\ x}\left\{\sum_{i=0}^{r}\left[\left(\frac{r!}{i! \cdot (r-i)!}\right) \cdot x^{r-i} \cdot (-\hat{M}_1\langle p\rangle)^i\right] \cdot p(x)\right\}$$

$$= \sum_{i=0}^{r}\left[(-1)^i \cdot \left(\frac{r!}{i! \cdot (r-i)!}\right) \cdot (\hat{M}_1\langle p\rangle)^i \cdot \sum_{all\ x}[x^{r-i} \cdot p(x)]\right]$$

$$= \sum_{i=0}^{r}\left[(-1)^i \cdot \left(\frac{r!}{i! \cdot (r-i)!}\right) \cdot (\hat{M}_1\langle p\rangle)^i \cdot \hat{M}_{r-i}\langle p\rangle\right] \tag{A.3}$$

Equation (A.3) gives the general relationship between the central and the absolute moments.

Furthermore, the skewness and the kurtosis of $p(x)$ are defined as:

$$\text{Skewness: } s\langle p\rangle = \sum_{all\ x}\left[\left(\frac{x-\mu\langle p\rangle}{\sigma\langle p\rangle}\right)^3 \cdot p(x)\right] \quad (A.4)$$

$$\text{Kurtosis: } k\langle p\rangle = \sum_{all\ x}\left[\left(\frac{x-\mu\langle p\rangle}{\sigma\langle p\rangle}\right)^4 \cdot p(x)\right] \quad (A.5)$$

where $\sigma\langle p\rangle = \sqrt{v\langle p\rangle}$ is the standard deviation of $p(x)$. It is obvious that

$$\widetilde{M}_3\langle p\rangle = (\sigma\langle p\rangle)^3 \cdot s\langle p\rangle \quad (A.6)$$

$$\widetilde{M}_4\langle p\rangle = (\sigma\langle p\rangle)^4 \cdot k\langle p\rangle \quad (A.7)$$

Based on the above definitions, we have

$$\mu\langle p\rangle = \hat{M}_1\langle p\rangle \quad (A.8)$$

$$v\langle p\rangle = \widetilde{M}_2\langle p\rangle = \hat{M}_2\langle p\rangle - (\hat{M}_1\langle p\rangle)^2 \quad (A.9)$$

$$\sigma\langle p\rangle = \sqrt{V\langle p\rangle} = \sqrt{\widetilde{M}_2\langle p\rangle} = \sqrt{M_2\langle p\rangle - (M_1\langle p\rangle)^2} \quad (A.10)$$

$$s\langle p\rangle = \frac{\widetilde{M}_3\langle p\rangle}{(\sigma\langle p\rangle)^3} = \frac{1}{[\hat{M}_2\langle p\rangle - (\hat{M}_1\langle p\rangle)^2]^{3/2}} \cdot [\hat{M}_3\langle p\rangle - 3\cdot\hat{M}_1\langle p\rangle\cdot\hat{M}_2\langle p\rangle + 2\cdot(\hat{M}_1\langle p\rangle)^3] \quad (A.11)$$

$$k\langle p\rangle = \frac{\widetilde{M}_4\langle p\rangle}{(\sigma\langle p\rangle)^4} = \frac{1}{[\hat{M}_2\langle p\rangle - (\hat{M}_1\langle p\rangle)^2]^2}$$

$$\cdot[\hat{M}_4\langle p\rangle - 4\cdot\hat{M}_1\langle p\rangle\cdot\hat{M}_3\langle p\rangle + 6\cdot(\hat{M}_1\langle p\rangle)^2\cdot\hat{M}_2\langle p\rangle - 3\cdot(\hat{M}_1\langle p\rangle)^4] \quad (A.12)$$

Since

$$\widetilde{M}_2\langle p\rangle = \sum_{all\ x}[(x-\mu\langle p\rangle)^2 \cdot p(x)]$$

$$= \hat{M}_2\langle p\rangle - (\hat{M}_1\langle p\rangle)^2 \quad (A.13)$$

$$\widetilde{M}_3\langle p\rangle = \sum_{all\ x}[(x-\mu\langle p\rangle)^3 \cdot p(x)]$$

$$= \hat{M}_3\langle p\rangle - (\hat{M}_1\langle p\rangle)^3 - 3\cdot\hat{M}_1\langle p\rangle\cdot[\hat{M}_2\langle p\rangle - (\hat{M}_1\langle p\rangle)^2] \quad (A.14)$$

$$\widetilde{M}_4\langle p\rangle = \sum_{all\ x}[(x-\mu\langle p\rangle)^4 \cdot p(x)]$$

$$= \hat{M}_4\langle p\rangle - (\hat{M}_1\langle p\rangle)^4 - 4\cdot\hat{M}_1\langle p\rangle\cdot[\hat{M}_3\langle p\rangle - (\hat{M}_1\langle p\rangle)^3]$$

$$+6\cdot(\hat{M}_1\langle p\rangle)^2\cdot[\hat{M}_2\langle p\rangle-(\hat{M}_1\langle p\rangle)^2] \tag{A.15}$$

we have

$$\hat{D}_2\langle p\rangle=\hat{M}_2\langle p\rangle-(\hat{M}_1\langle p\rangle)^2$$

$$=\widetilde{M}_2\langle p\rangle=v\langle p\rangle=(\sigma\langle p\rangle)^2 \tag{A.16}$$

$$\hat{D}_3\langle p\rangle=\hat{M}_3\langle p\rangle-(\hat{M}_1\langle p\rangle)^3$$

$$=\widetilde{M}_3\langle p\rangle+3\cdot\hat{M}_1\langle p\rangle\cdot\widetilde{M}_2\langle p\rangle$$

$$=(\sigma\langle p\rangle)^3\cdot s\langle p\rangle+3\cdot\mu\langle p\rangle\cdot(\sigma\langle p\rangle)^2$$

$$=(\sigma\langle p\rangle)^2\cdot[3\cdot\mu\langle p\rangle+\sigma\langle p\rangle\cdot s\langle p\rangle] \tag{A.17}$$

$$\hat{D}_4\langle p\rangle=\hat{M}_4\langle p\rangle-(\hat{M}_1\langle p\rangle)^4$$

$$=\widetilde{M}_4\langle p\rangle+4\cdot\hat{M}_1\langle p\rangle\cdot[\hat{M}_3\langle p\rangle-(\hat{M}_1\langle p\rangle)^3]-6\cdot(\hat{M}_1\langle p\rangle)^2\cdot[\hat{M}_2\langle p\rangle-(\hat{M}_1\langle p\rangle)^2]$$

$$=(\sigma\langle p\rangle)^4\cdot k\langle p\rangle+4\cdot\mu\langle p\rangle\cdot(\sigma\langle p\rangle)^2\cdot[3\cdot\mu\langle p\rangle+\sigma\langle p\rangle\cdot s\langle p\rangle]-6\cdot(\mu\langle p\rangle)^2\cdot(\sigma\langle p\rangle)^2$$

$$=(\sigma\langle p\rangle)^2\cdot[6\cdot(\mu\langle p\rangle)^2+4\cdot\mu\langle p\rangle\cdot\sigma\langle p\rangle\cdot s\langle p\rangle+(\sigma\langle p\rangle)^2\cdot k\langle p\rangle] \tag{A.18}$$

$\blacksquare$

Annex B. Some properties of the Gamma function

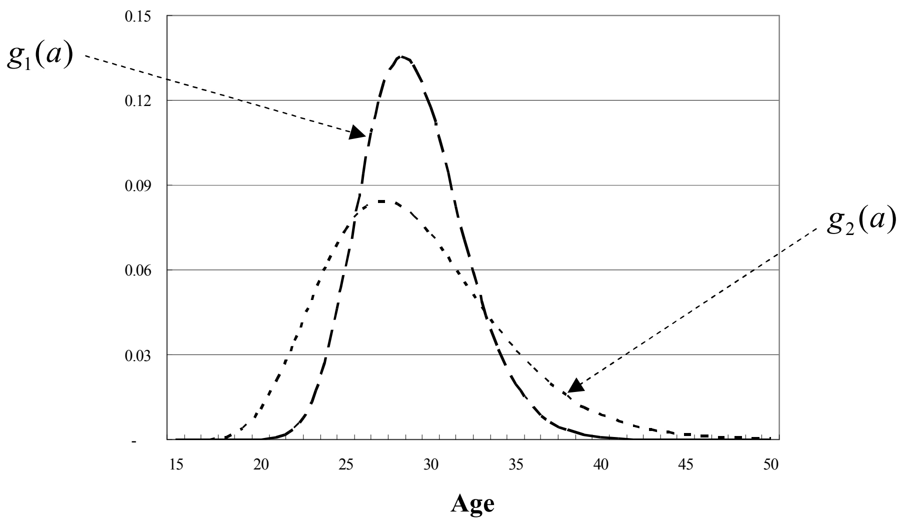
The general formula for the Gamma function is given by

$$g(a)=\begin{cases}K\cdot(a-a_0)^A\cdot e^{-B\cdot(a-a_0)}, & \text{when } a\geq a_0\\ 0, & \text{when } a<a_0\end{cases} \tag{B.1}$$

where (i) a_0 is a constant, representing the start point of the curve, (ii) A ($A>0$) and B ($B>0$) are constants, which determine the shape of the curve, and (iii) K ($K>0$) is a coefficient, which ensures that $\int_{a_0}^{\infty} g(a)da=1$, i.e. $K\cdot\int_{a_0}^{\infty}[(a-a_0)^A\cdot e^{-B\cdot(a-a_0)}]da=1$. Therefore, $K=1\Big/\int_{a_0}^{\infty}[(a-a_0)^A\cdot e^{-B\cdot(a-a_0)}]da$ once a_0, A and B are known. It is obvious that $g(a)\geq 0$, $-\infty<a<\infty$.

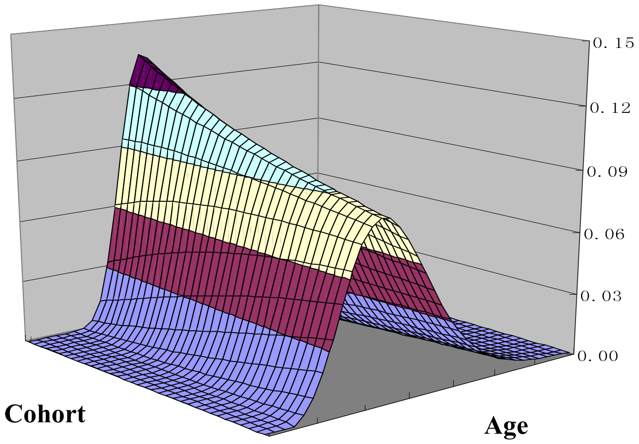
The following graph shows two concrete examples of curve $g(a)$: $g_1(a)$ ($a_0=15$, $\mu\langle g_1\rangle=29$, $\sigma\langle g_1\rangle=3$) and $g_2(a)$ ($a_0=15$, $\mu\langle g_2\rangle=29$, $\sigma\langle g_2\rangle=5$).

Figure B.1. Examples of Gamma distribution



The following graph shows in three-dimensional form the surface of $g(a)$, using $g_1(a)$ as the start curve and $g_2(a)$ as the end curve. The curves between $g_1(a)$ and $g_2(a)$ are derived based on linear interpolation vis-à-vis the standard deviations.

Figure B.2. Three-dimensional presentation of $g(a)$ surface



The n^{th} absolute moment (about zero or origin) of $g(a)$ is defined as $\hat{M}_n\langle g\rangle = \int_{a_0}^{\infty}[a^n \cdot g(a)]da$, where n is a non-negative integer. Let $u = a - a_0$, then we have $u \geq 0$, $a = u + a_0$, and $da = du$. Hence,

$$\hat{M}_n\langle g\rangle = \int_{a_0}^{\infty}[a^n \cdot g(a)]da = K \cdot \int_0^{\infty}[(u+a_0)^n \cdot u^A \cdot e^{-B\cdot u}]du$$

$$= K \cdot \int_0^{\infty}\left[\sum_{i=0}^{n}\left(\frac{n!}{i!\cdot(n-i)!}\cdot u^{n-i}\cdot a_0^i\right)\cdot u^A \cdot e^{-B\cdot u}\right]du$$

$$= K \cdot \sum_{i=0}^{n}\left[a_0^i \cdot \frac{n!}{i!\cdot(n-i)!}\cdot \int_0^{\infty}(u^{A+(n-i)}\cdot e^{-B\cdot u})du\right] \qquad \text{(B.2)}$$

Let $U_{n-i} = \int_0^{\infty}(u^{A+(n-i)}\cdot e^{-B\cdot u})du$, then it is obvious that $K \cdot U_0 = K \cdot \int_0^{\infty}(u^A \cdot e^{-B\cdot u})du = 1$ or $U_0 = 1/K$.

Further, we have

$$U_{n-i} = -\frac{1}{B}\cdot\int_0^{\infty}u^{A+(n-i)}d(e^{-B\cdot u}) = -\frac{1}{B}\cdot\left[(u^{A+(n-i)}\cdot e^{-B\cdot u})\Big|_0^{\infty} - \int_0^{\infty}e^{-B\cdot u}d(u^{A+(n-i)})\right]$$

$$= -\frac{1}{B}\cdot\left[\lim_{u\to\infty}(u^{A+(n-i)}\cdot e^{-B\cdot u}) - [A+(n-i)]\cdot\int_0^{\infty}(u^{A+((n-i)-1)}\cdot e^{-B\cdot u})du\right]$$

$$= -\frac{1}{B}\cdot\left[\lim_{u\to\infty}(u^{A+(n-i)}\cdot e^{-B\cdot u}) - [A+(n-i)]\cdot U_{(n-i)-1}\right] \qquad \text{(B.3)}$$

By Taylor series expansion, we have

$$e^{B\cdot u} = 1 + \frac{B\cdot u}{1!} + \frac{(B\cdot u)^2}{2!} + \frac{(B\cdot u)^3}{3!} + \cdots + \frac{(B\cdot u)^i}{i!} + \cdots \qquad \text{(B.4)}$$

Taking a positive integer *m*, such that $m > A + n$, then we have $e^{B\cdot u} > \frac{(B\cdot u)^m}{m!}$ or equivalently $\frac{1}{e^{B\cdot u}} < \frac{m!}{(B\cdot u)^m} = \left(\frac{m!}{B^m}\right)\cdot\left(\frac{1}{u^m}\right)$. Since $0 \le i \le n$, it follows that $A \le A + (n-i) \le A + n$. Therefore,

$$0 < u^{A+(n-i)} \cdot e^{-B\cdot u} = \frac{u^{A+(n-i)}}{e^{B\cdot u}} < \left(\frac{m!}{B^m}\right)\cdot\left(\frac{1}{u^{m-(A+(n-i))}}\right) < \left(\frac{m!}{B^m}\right)\cdot\left(\frac{1}{u^{m-(A+n)}}\right) \qquad \text{(B.5)}$$

Since $\lim_{u\to\infty}\left[\left(\frac{m!}{B^m}\right)\cdot\left(\frac{1}{u^{m-(A+n)}}\right)\right] = 0$, it follows from the squeeze theorem of limit that $\lim_{u\to\infty}(u^{A+(n-i)} \cdot e^{-B\cdot u}) = 0$, $i = 0, 1, 2, \ldots, n$. Therefore, equation (B.3) becomes

$$U_{n-i} = \frac{A + (n-i)}{B} \cdot U_{(n-i)-1}, \quad i = 0, 1, 2, \ldots, n-1 \qquad \text{(B.6)}$$

By mathematical induction, it can be proved that $K \cdot U_{n-i} = \left[\prod_{j=1}^{n-i}(A + j)\right]\Big/ B^{n-i}$, $i = 0, 2, \ldots, n-1$ (or equivalently, $K \cdot U_r = \left[\prod_{j=1}^{r}(A + j)\right]\Big/ B^{r}$ $r = 1, 2, \ldots, n$). Therefore, equation (B.2) can be rewritten as

$$\hat{M}_n\langle g\rangle = \sum_{i=0}^{n}\left[a_0^i \cdot \frac{n!}{i!\cdot(n-i)!} \cdot (K \cdot U_{n-i})\right] \qquad \text{(B.7)}$$

Specially, we have

$$\hat{M}_1\langle g\rangle = \sum_{i=0}^{1}\left[a_0^i \cdot \frac{1!}{i!\cdot(1-i)!} \cdot (K \cdot U_{1-i})\right] = a_0 \cdot (K \cdot U_0) + K \cdot U_1$$

$$= a_0 + \frac{A+1}{B} \qquad \text{(B.8)}$$

$$\hat{M}_2\langle g\rangle = \sum_{i=0}^{2}\left[a_0^i \cdot \frac{2!}{i!\cdot(2-i)!} \cdot (K \cdot U_{2-i})\right]$$

$$= a_0^2 \cdot (K \cdot U_0) + 2 \cdot a_0 \cdot (K \cdot U_1) + K \cdot U_2$$

$$= a_0^2 + 2 \cdot a_0 \cdot \frac{A+1}{B} + \frac{(A+1)\cdot(A+2)}{B^2} \tag{B.9}$$

$$\hat{M}_3\langle g\rangle = \sum_{i=0}^{3}\left[a_0^i \cdot \frac{3!}{i!\cdot(3-i)!}\cdot(K\cdot U_{3-i})\right]$$

$$= a_0^3 \cdot (K\cdot U_0) + 3\cdot a_0^2 \cdot (K\cdot U_1) + 3\cdot a_0 \cdot (K\cdot U_2) + K\cdot U_3$$

$$= a_0^3 + 3\cdot a_0^2 \cdot (K\cdot U_1) + 3\cdot a_0 \cdot (K\cdot U_2) + K\cdot U_3 \tag{B.10}$$

$$\hat{M}_4\langle g\rangle = \sum_{i=0}^{4}\left[a_0^i \cdot \frac{4!}{i!\cdot(4-i)!}\cdot(K\cdot U_{4-i})\right]$$

$$= a_0^4 \cdot (K\cdot U_0) + 4\cdot a_0^3 \cdot (K\cdot U_1) + 6\cdot a_0^2 \cdot (K\cdot U_2) + 4\cdot a_0 \cdot (K\cdot U_3) + K\cdot U_4$$

$$= a_0^4 + 4\cdot a_0^3 \cdot (K\cdot U_1) + 6\cdot a_0^2 \cdot (K\cdot U_2) + 4\cdot a_0 \cdot (K\cdot U_3) + K\cdot U_4 \tag{B.11}$$

Therefore, we have

(i) *Mean of* $g(a)$

$$\mu\langle g\rangle = \int_{a_0}^{\infty}[a\cdot g(a)]da = \hat{M}_1\langle g\rangle = a_0 + \frac{A+1}{B} \tag{B.12}$$

(ii) *Variance of* $g(a)$

$$v\langle g\rangle = \int_{a_0}^{\infty}[(a-\mu\langle g\rangle)^2 \cdot g(a)]da = \hat{M}_2\langle g\rangle - (\hat{M}_1\langle g\rangle)^2$$

$$= \frac{(A+1)\cdot(A+2)}{B^2} - \left(\frac{A+1}{B}\right)^2 = \frac{A+1}{B^2} \tag{B.13}$$

(iii) *Standard deviation of* $g(a)$

$$\sigma\langle g\rangle = \sqrt{v\langle g\rangle} = \frac{\sqrt{A+1}}{B} \tag{B.14}$$

(iv) *Skewness of* $g(a)$

$$s\langle g\rangle = \int_{a_0}^{\infty}\left[\left(\frac{a-\mu\langle g\rangle}{\sigma\langle g\rangle}\right)^3 \cdot g(a)\right]da$$

$$= \frac{1}{(\sigma\langle g\rangle)^3} \cdot [\hat{M}_3\langle g\rangle - 3 \cdot \hat{M}_1\langle g\rangle \cdot \hat{M}_2\langle g\rangle + 2 \cdot (\hat{M}_1\langle g\rangle)^3]$$

$$= \frac{1}{(\sigma\langle g\rangle)^3} \cdot \left[\frac{(A+1)\cdot(A+2)\cdot(A+3)}{B^3} - 3 \cdot \frac{(A+1)^2\cdot(A+2)}{B^3} + 2 \cdot \left(\frac{A+1}{B}\right)^3 \right]$$

$$= \frac{1}{(\sigma\langle g\rangle)^3} \cdot 2 \cdot \left(\frac{A+1}{B^3}\right) = 2 \cdot \left(\frac{\sqrt{A+1}}{A+1}\right) \tag{B.15}$$

(note that $\sigma\langle g\rangle \cdot B = \sqrt{A+1}$, therefore $(\sigma\langle g\rangle \cdot B)^3 = (A+1)^{3/2}$)

(v) *Kurtosis of* $g(a)$

$$k\langle g\rangle = \int_{a_0}^{\infty} \left[\left(\frac{a-\mu\langle g\rangle}{\sigma\langle g\rangle} \right)^4 \cdot g(a) \right] da$$

$$= \frac{1}{(\sigma\langle g\rangle)^4} \cdot [\hat{M}_4\langle g\rangle - 4 \cdot \hat{M}_1\langle g\rangle \cdot \hat{M}_3\langle g\rangle + 6 \cdot (\hat{M}_1\langle g\rangle)^2 \cdot \hat{M}_2\langle g\rangle - 3 \cdot (\hat{M}_1\langle g\rangle)^4]$$

$$= \frac{1}{(\sigma\langle g\rangle)^4} \cdot \left[\frac{(A+1)\cdot(A+2)\cdot(A+3)\cdot(A+4)}{B^4} - 4 \cdot \frac{(A+1)^2\cdot(A+2)\cdot(A+3)}{B^4} \right.$$

$$\left. + 6 \cdot \frac{(A+1)^3\cdot(A+2)}{B^4} - 3 \cdot \left(\frac{A+1}{B}\right)^4 \right]$$

$$= \frac{1}{(\sigma\langle g\rangle)^4} \cdot \left(\frac{A+1}{B^4}\right) \cdot 3 \cdot (A+3) = 3 \cdot \left(\frac{A+3}{A+1}\right) \tag{B.16}$$

(note that $\sigma\langle g\rangle \cdot B = \sqrt{A+1}$, therefore $(\sigma\langle g\rangle \cdot B)^4 = (A+1)^2$)

From equation (B.12), we have $\frac{A+1}{B} = \mu\langle g\rangle - a_0$. Therefore, from equation (B.13). we have

$$B = \left(\frac{A+1}{B}\right) \Big/ v\langle g\rangle = \frac{\mu\langle g\rangle - a_0}{v\langle g\rangle} \tag{B.17}$$

It then follows that

$$A = B \cdot (\mu\langle g\rangle - a_0) - 1 = \frac{(\mu\langle g\rangle - a_0)^2}{v\langle g\rangle} - 1 \tag{B.18}$$

The first derivative of $g(a)$ with respect to *a* is as follows:

$$[g(a)]'_a = K \cdot (a - a_0)^{A-1} \cdot e^{-B \cdot (a - a_0)} \cdot [A - B \cdot (a - a_0)] \quad \text{(B.19)}$$

By setting $[g(a)]'_a = 0$, we obtain $a = a_0 + \frac{A}{B}$ (ignore $a = a_0$). This tells us that the curve $\gamma(a)$ attains its maximum at $a_{\max} = a_0 + \frac{A}{B}$. Since $\mu\langle g \rangle - a_{\max} = 1/B > 0$, we have $\mu\langle g \rangle > a_{\max}$. This implies that curve $g(a)$ is always positively skewed (i.e. with the longer tail always on the right-hand side of the curve).

In terms of data fitting using the gamma function, the following method can be used. Taking the natural logarithm on both sides of equation (B.1), we obtain

$$\ln[g(a)] = \ln(K) + A \cdot \ln(a - a_0) - B \cdot (a - a_0) \quad \text{(B.20)}$$

where $a > a_0$. Let $y = \ln[g(a)]$, $\theta_0 = \ln(K)$, $\theta_1 = A$, $\theta_2 = -B$, $x_1 = \ln(a - a_0)$ and $x_2 = a - a_0$, then equation (B.16) becomes:

$$y = \theta_0 + \theta_1 \cdot x_1 + \theta_2 \cdot x_2 \quad \text{(B.21)}$$

By applying bivariate linear regression to equation (B.18), we can obtain the estimates for coefficients θ_0, θ_1 and θ_2 (denoted as $\hat{\theta}_0$, $\hat{\theta}_1$ and $\hat{\theta}_2$, respectively). Then we have $K = e^{\hat{\theta}_0}$, $A = \hat{\theta}_1$, and $B = -\hat{\theta}_2$.

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