Application of Tangent Vector Fields on the Log Mortality Surface to Mortality Projection for Japan

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Introduction

The Lee-Carter(LC) model, which is considered as a kind of relational models, is now widely used for mortality projection. Relational models, which possess combined features of the tabular approach of model life tables and the mathematical approach, express the mortality age pattern by a *standard* mortality pattern and some parameters that mathematically describe the *deviation* from the standard pattern. Since the LC model is usually used for time series modeling, the *deviation* from the *standard* pattern is based on the mortality change over time.

Normally, more attention has been paid to the resulted mortality age pattern than to the change in mortality curve over time in the context of modeling. In this study, we put special emphasis on the change in mortality curve, and propose a new *shift*-type model (the LD model) and a novel method to construct mortality projection model applying the *tangent vector fields on the log mortality surface*, which serves as a flexible tool to describe the change in mortality curve to any directions.

1 Lee-Carter Model and Its Differential Form

Let us denote $\mu_{x,t}$ as the hazard function for exact age x at time t, and $y = \lambda_{x,t} = \log \mu_{x,t}$ as the log hazard function of mortality. Then, the set $S = \{(x,t,y)|y = \lambda_{x,t}\}$ determines a surface in $\mathbb{R}^3$, called the *log mortality surface*. We assume that $\lambda_{x,t}$ is a smooth continuous function with respect to x and t.

The LC model is expressed by the following formula (Lee and Carter 1992).(We call it the *normal form*.)

$$\lambda_{x,t} = \log \mu_{x,t} = a_x + k_t b_x$$

where a_x is a standard age pattern of mortality.

Taking a partial derivative by time t, we obtain the following relationship.(We call it the *differential form*.)

$$\rho_{x,t} = -\frac{dk_t}{dt} b_x = -k'_t b_x$$

Note that the differential form is another way to define the LC model. Suppose $\rho_{x,t}$ is a product of two parameters $-k'_t$ and b_x, then $\lambda_{x,t}$ satisfies the normal form of the LC model. In this study, we prefer working with the differential form since we put special emphasis on the change in mortality curve.

2 The Linear Difference (LD) Model

We have already seen in Ishii (2008) that the recent adult mortality improvement in Japan could be modeled better by the *shift*-type model than by the *decline*-type model such as the LC model. In this section, we define a new *shift*-type model of *adult* mortality, which we call the Linear Difference (LD) model.

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First, note that considering the *shift*-type model for the log mortality rate is identical to considering the *decline*-type model for the *inverse* function of the log mortality rates as shown in Figure 1 and 2. Therefore, we work with the inverse function of log mortality to define the LD model. Let us assume that $\lambda_{x,t}$ is a strictly monotonic increasing with respect to x for each t so that $\lambda_t(x) = \lambda_{x,t}$ may have an inverse function $\nu_t(y)$ for each t. Then, we can define the function $\nu_{y,t} = \nu_t(y)$ which leads to another representation of log mortality surface for the adult mortality.

Figure 1 Log Mortality Rates

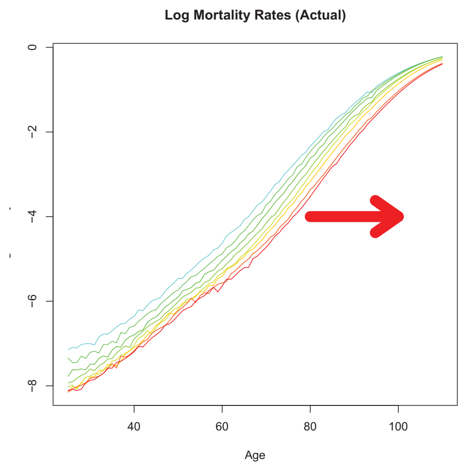
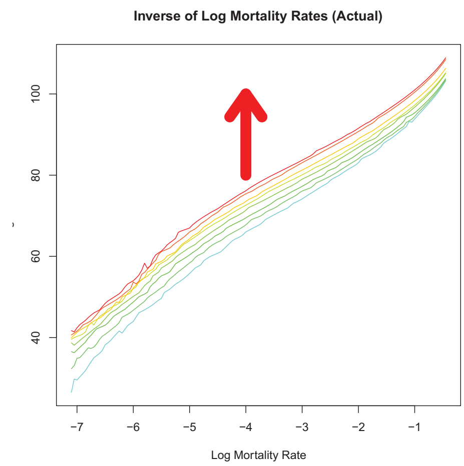


Figure 2 Inverse of Log Mortality Rates)



As we defined the mortality improvement rate $\rho_{x,t}$ for the log mortality, we can define a similar function for the inverse log mortality, i.e. $\tau_{y,t}$: the force of age increase by $\tau_{y,t} \underset{\text{def}}{=} \frac{\partial \nu_{y,t}}{\partial t}$. Using this function, we define the LD model on condition that $\tau_{y,t}$ is a linear function of x for each t, i.e. $\tau_{y,t} = k'_t + c'_t x$.

We can show that this property holds for the two parameter logistic model, which implies a close relationship between the two models. We can observe from Figure 3 and 4 that the LD model fits quite well to the actual adult female mortality for Japan in Human Mortality Database.

Figure 3 Inverse Mortality Rates (Actual and Model, LD)

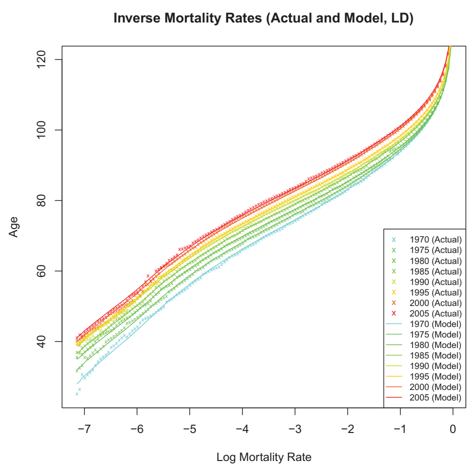
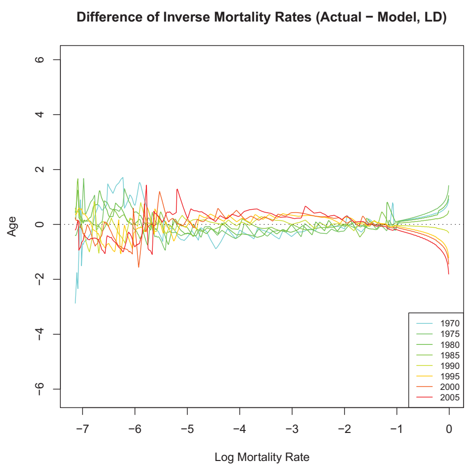


Figure 4 Difference of Inverse Mortality Rates (Actual and Model, LD)



3 Tangent Vector Fields on the Log Mortality Surface

In the previous section, we proposed the LD model for the *adult* mortality. However, we need an entire age model for mortality projection. Here, we propose a novel method to construct an entire age model with the LD model structure for adult mortality and the LC model for juvenile applying *tangent vector field* approach.

We begin with a stylized example of the change in mortality curves shown in Figure 5. Now, we are going to use the LD model for the adult mortality, whose direction of the mortality improvements expressed by the age-increases shown in the red arrows. On the other hand, the mortality improvements in the juvenile mortality are well-modeled by the *decline*- type models, such as the LC model whose mortality improvements shown in the blue arrows.

Here, the arrows express the directions for which the points on the log mortality curves are heading. Mathematically, these arrows are formulated using *tangent vector fields* on the log mortality surface.

In Figure 6, the vectors

$$\boldsymbol{\rho} = \boldsymbol{\rho}(x_0, t_0, y_0) = (0, 1, -\rho_{x_0,t_0})$$

$$\boldsymbol{\tau} = \boldsymbol{\tau}(x_0, t_0, y_0) = (\tau_{y_0,t_0}, 1, 0)$$

are tangent vectors on S. Each tangent vector defines a tangent vector field on S.

On the other hand, if we have vectors $\boldsymbol{\xi}$ that determines the direction of the mortality change for each t, then we can construct a log mortality surface whose tangent vector field is $\boldsymbol{\xi}$.

For example, the vector $\boldsymbol{\rho}(x, t, y) = (0, 1, -\rho_{x,t})$ with $\rho_{x,t} = -k'_t b_x$ induces a log mortality surface that follows the LC model. Similarly, the vector $\boldsymbol{\tau}(x, t, y) = (\tau_y, 1, 0)$ with $\tau_{y,t} = k'_t + c'_t x$ induces a log mortality surface that follows the LD model.

Using a weight function$w(x, t)$ which takes 0 on young age and 1 on old age, we can define a new tangent vector field $\boldsymbol{\xi}$ as follows.

$$\boldsymbol{\xi} = (1 - w(x, t))\boldsymbol{\rho}(x, t, y) + w(x, t)\boldsymbol{\tau}(x, t, y)$$

We can define a log mortality surface that has the above tangent vector field. We call it TVF (Tangent Vector Fields) model here.

Figure 7 shows the example of the tangent vector fields on the log mortality surface corresponding to the LC (the blue arrows) and LD models (the red arrows). We can define the tangent vector field for the LC model in the entire age, whereas the one for the LD model is defined only in adult mortality.

Figure 8 compares the estimated mortality rates by LC, LD and TVF models. We can observe that the mortality rates estimated by the TVF model correspond to the LC model in young age and to the LD model in old age.

References

Human Mortality Database. University of California, Berkeley (USA) and Max Planck Institute for Demographic Research (Germany). Available at www.mortality.org or www.humanmortality.de.

Ishii, F. (2008) “Mortality Projection Model for Japan with Age-Shifting Structure”, Paper presented at 2008 Annual Meeting of Population Association of America (New Orleans).

Lee, R. and L. Carter (1992) “Modeling and Forecasting U.S. Mortality”, *Journal of the American Statistical Association*, Vol. 87, No. 419, pp. 659–675.

Figure 5　Change in the Mortality Curves

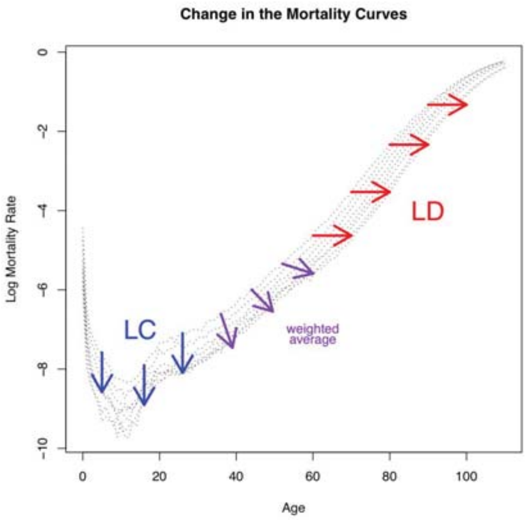


Figure 6　Tangent Vectors on S

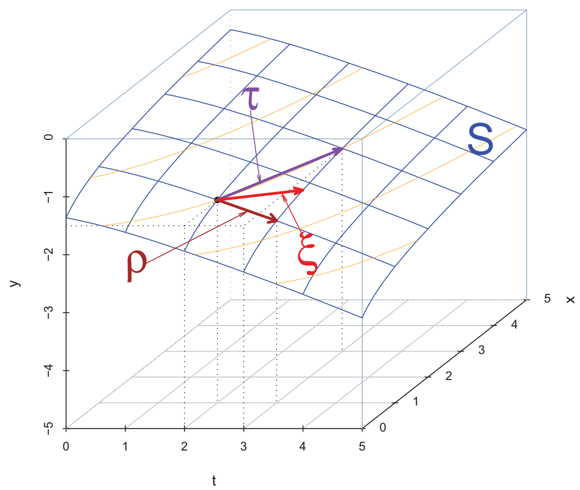


Figure 7　Example of a Construction of a Tangent Vector Field

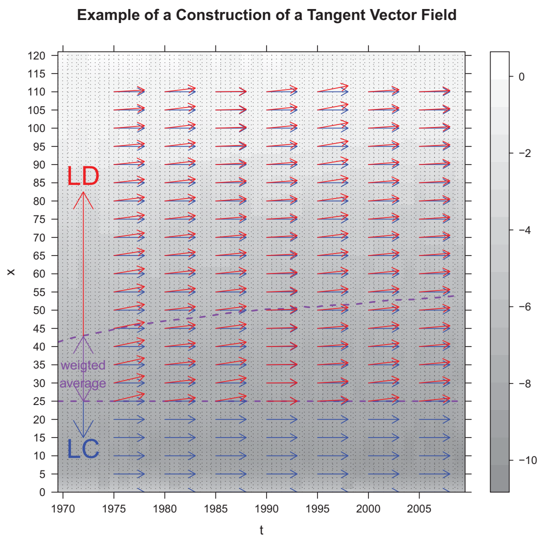


Figure 8　Estimated Mortality Rates